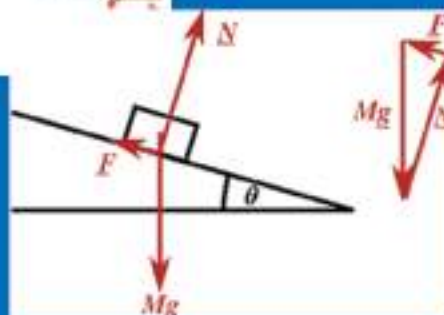
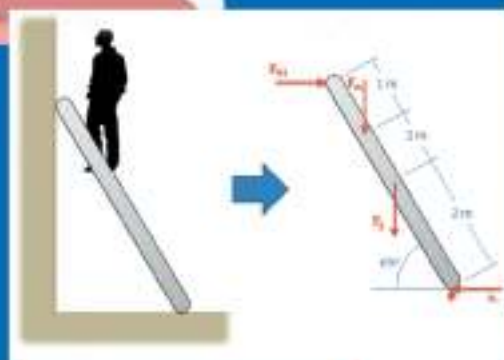
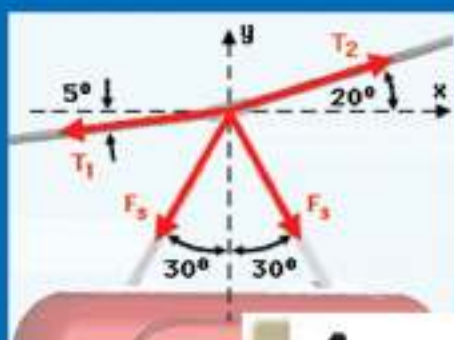




G.C.E. Advanced Level
Combined Mathematics
STATICS - I
Additional Reading Book

(Prepared According to the New syllabus Implemented From 2017)



Department of Mathematics
National Institute of Education
Maharagama
Sri Lanka
www.nie.lk

G.C.E. Advanced Level

Combined Mathematics

STATICS - I

Additional Reading Book

Department of Mathematics
Faculty of Science and Technology
National Institute of Education
Maharagama

Combined Mathematics
Statics - I
Additional Reading Book

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Message from the Director General

Department of Mathematics of National Institute of Education time to time implements many different activities to develop the mathematics education. The publication of this book is a milestone which was written in the name of **“Statics - Part I, Statics - Part II”**.

After learning of grade 12 and 13 syllabus, teachers should have prepared the students for the General Certificate of Education (Advanced Level) which is the main purpose of them. It has not enough appropriate teaching - learning tools for the proper utilization. It is well known to all, most of the instruments available in the market are not appropriate for the use and it has not enough quality in the questions. Therefore **“Statics - Part I, Statics - Part II”**. book was prepared by the Department of Mathematics of National Institute of Education which was to change of the situation and to ameliorate the students for the examination. According to the syllabus the book is prepared for the reference and valuable book for reading. Worked examples are included which will be helpful to the teachers and the students.

I kindly request the teachers and the students to utilize this book for the mathematics subjects' to enhance the teaching and learning process effectively. My gratitude goes to Aus Aid project for sponsoring and immense contribution of the internal and external resource persons from the Department of Mathematics for toil hard for the book of **“Statics - Part I, Statics - Part II”**.

Dr. (Mrs). T. A. R. J. Gunasekara

Director General

National Institute of Education.

Message from the Director

Mathematics holds a special place among the G.C.E. (A/L) public examination prefer to the mathematical subject area. The footprints of the past history record that the country's as well as the world's inventor's spring from the mathematical stream.

The aim and objectives of designing the syllabus for the mathematics stream is to prepare the students to become experts in the Mathematical, Scientific and Technological world.

From 2017 the Combined Mathematics syllabus has been revised and implemented. To make the teaching - learning of these subjects easy, the Department of Mathematics of National Institute of Education has prepared **Statics - Part 1 and Part 11** as the supplementary reading books. There is no doubt that the exercises in these books will measure their achievement level and will help the students to prepare themselves for the examination. By practicing the questions in these books the students will get the experience of the methods of answering the questions. Through the practice of these questions, the students will develop their talent, ability, skills and knowledge. The teachers who are experts in the subject matter and the scholars who design the syllabus, pooled their resources to prepare these supplementary reading books. While preparing these books, much care has been taken that the students will be guided to focus their attention from different angles and develop their knowledge. Besides, the books will help the students for self-learning.

I sincerely thank the Director General for the guidance and support extended and the resource personnel for the immense contribution. I will deeply appreciate any feedback that will shape the reprint of the books.

Mr. K. R. Pathmasiri

Director

Department of Mathematics

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Preface

This book is being prepared for the students of Combined Mathematics G.C.E.A/L to get familiar with the subject area of Statics. It is a supplementary book meant for the students to get practice in answering the questions for self learning. The teachers and the students are kindly invited to understand, it is not a bunch of model questions but a supplementary to encourage the students towards self learning and to help the students who have missed any area in the subject matter to rectify them.

The students are called upon to pay attention that after answering the questions in worked examples by themselves, they can compare their answers with the answers given in the book. But it is not necessary that all the steps have taken to arrive at the answers should tally with the steps mentioned in the book's answers given in this book are only a guide.

Statics - Part I is released in support of the revised syllabus - 2017. The book targets the students who will sit for the GCE A/L examination – 2019 onwards. The Department of Mathematics of National Institute of Education already released **Practice Questions and Answers** book and it is being proceeded by the “**Statics I**”. There are other books soon be released with the questions taken Unit wise “**Questions bank**” and “**Statics - Part II**”.

We shall deeply appreciate your feedback that will contribute to the reprint of this book.

Mr.S.Rajendram

Project Leader

Grade 12, 13 Maths

National Institute of Education.

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1.0 Vectors

1.1 Scalar quantities

Quantities which can be entirely determined by numbers with appropriate units are called scalar quantities.

Distance, time, mass, volume, temperature are scalar quantities.

Further, two quantities of the same kind, when added will give another quantity of the same kind.

Examples:

Mass is 10 kg; Temperature is 27°C , Time is 20 s. Length is 2 m; Area is 5 m^2 , Volume is 4 m^3 , capacity is 2 l, speed is 5 m s^{-1} . The numerical parts of the above examples without the units are called **Scalars**.

There are also quantities which cannot be described fully by magnitude (with units) alone but which can be known fully by magnitude and direction. For example,

- i) A ship is travelling with a speed of 15 km h^{-1} due north.
- ii) A force 20 newton act on a particle vertically downwards.

Study on vectors was first focused in middle of 19th century. In the recent past “Vectors” has become an indispensable tool, used in the mathematical calculation of engineers, mathematician and physicists while physical and geometrical problems can be expressed concisely, by using vectors.

1.2 Vector quantities

Quantities which can be described completely by magnitudes (with units) and directions are called vector quantities.

Examples:

- i. Displacement due north is 5 m.
- ii. Velocity is 15 m s^{-1} due south east.
- iii. Weight is 30 N, Vertically downwards.
- iv. Force of 10 N inclined upwards 30° to the horizontal.

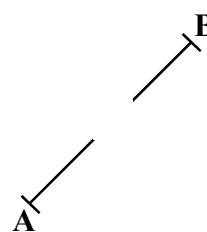
Vectors have both magnitude and direction.

1.3 Representation of vectors

There are two ways of representing vectors.

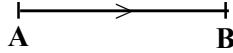
Geometrical Representation

A vector can be represented by a **directed line segment** \overrightarrow{AB}
The length of the line segment will give the magnitude of the vector and the arrow head on it denotes the direction. This is said to be the geometrical representation of a vector.



Example:

To denote a force of 4 N due east a straight line segment AB is drawn towards east where AB = 4 units. The direction of the force is denoted by the arrow from A to B as shown below.



Algebraic Representation

The vector \overrightarrow{AB} is denoted by a single algebraic symbol such as \underline{a} or \bar{a} . In some text books generally it is denoted by the symbol **a** in dark print.

1.4 Modulus of a vector

The magnitude of a vector is known as its modulus.

The modulus of a vector \overrightarrow{AB} (or \underline{a}) is denoted by $|\overrightarrow{AB}|$ or $|\underline{a}|$

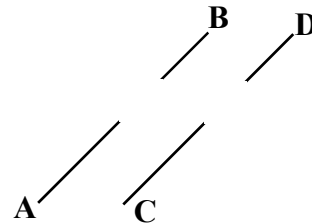
The modulus of a vector is always non-negative.

1.5 Equality of two vectors

If two vectors are equal in magnitude and are in the same direction they are called equal vectors.

The two vectors \overrightarrow{AB} ($=\underline{a}$) and \overrightarrow{CD} ($=\underline{b}$) are equal if and only if

- i) $|\overrightarrow{AB}| = |\overrightarrow{CD}|$
- ii) $AB \parallel CD$ and
- (iii) \overrightarrow{AB} and \overrightarrow{CD} are in the same direction.



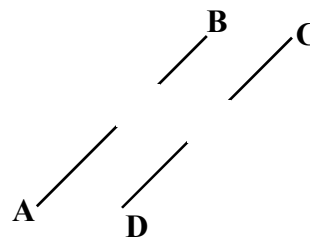
Note : Consider the vectors \overrightarrow{AB} and \overrightarrow{CD}

$AB = CD$ i.e $|\overrightarrow{AB}| = |\overrightarrow{CD}|$

$AB \parallel CD$

But they are not in the same direction.

Therefore $\overrightarrow{AB} \neq \overrightarrow{CD}$; $\underline{a} \neq \underline{b}$



1.6 Unit vector

A vector with unit magnitude is called unit vector. Given a vector \underline{a} , the unit vector in the direction of \underline{a} is $\frac{\underline{a}}{|\underline{a}|}$ is denoted by \hat{a} .

1.7 Zero vector (null vector)

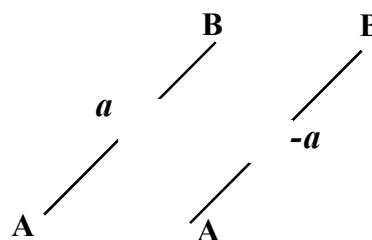
A vector with zero magnitude is called zero vector. It is denoted by $\mathbf{0}$. $|\mathbf{0}| = 0$ and its direction is arbitrary and is represented by a point.

1.8 Negative vector of a given vector

Given a vector \vec{AB} , the vector \vec{BA} is negative vector of \vec{AB} and is written $\vec{BA} = -\vec{AB}$.

If $\vec{AB} = \underline{a}$, then $\vec{BA} = -\underline{a}$

$$|\vec{AB}| = |\vec{BA}|, |\underline{a}| = |-\underline{a}|$$



1.9 Scalar multiple of a vector

When \underline{a} is a vector and λ is a scalar, then $\lambda\underline{a}$ is the product of the vector \underline{a} and scalar λ . Here λ should be considered under three cases namely when $\lambda > 0$, $\lambda = 0$ and $\lambda < 0$.

Case (i) $\lambda > 0$

Let $\vec{OA} = \underline{a}$,

Take a point B on OA.

(or produced OA) such that $OB = \lambda OA$

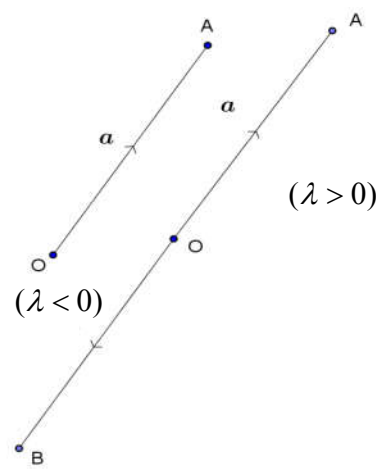
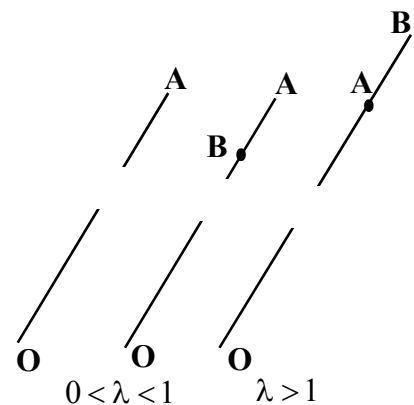
$$\vec{OB} = \lambda \vec{OA} = \lambda \underline{a}$$

(ii) When $\lambda = 0$, $\lambda\underline{a}$ is defined as the null vector.

$$\text{That is } \lambda \underline{a} = 0 \underline{a} = \mathbf{0}$$

(iii) $\lambda < 0$

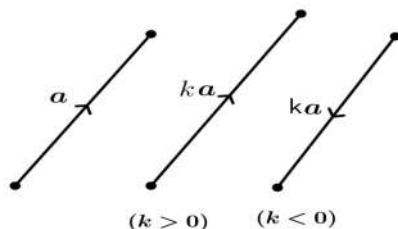
In this case $\lambda\underline{a}$ is a vector opposite to the direction of \underline{a} with a magnitude of $|\lambda|$ times OA. Choose a point B on AO produced such that $OB = |\lambda| OA$. Then $\vec{OB} = \lambda \underline{a}$.



1.10 Parallel vectors

Given a vector \underline{a} and $k\underline{a}$, $k\underline{a}$ is a vector parallel to \underline{a}

- (i) When $k > 0$, the vector $k\underline{a}$ is in the direction of \underline{a}
- (ii) When $k < 0$, the vector $k\underline{a}$ is opposite in direction to that of \underline{a} .



Two vectors \underline{a} and \underline{b} are said to be parallel. if $\underline{b} = \lambda\underline{a}$

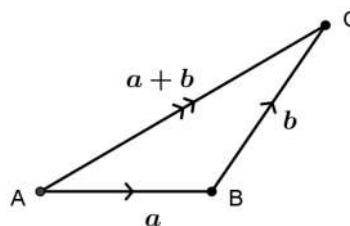
1.11 Vector addition

If two vectors \underline{a} and \underline{b} are represented by \overrightarrow{AB} and \overrightarrow{BC} respectively then the vector addition of \underline{a} and \underline{b} is represented by \overrightarrow{AC}

ie.
$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC}$$

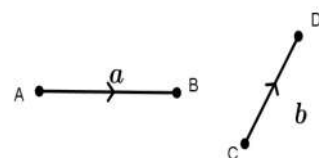
$$= \underline{a} + \underline{b}$$

This is called the triangle law of vector addition.



Let $\overrightarrow{AB} = \underline{a}$ and $\overrightarrow{CD} = \underline{b}$ be two vectors.

Draw a line segment PQ such that PQ = AB and PQ // AB.

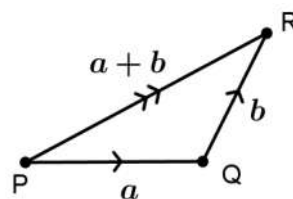


Draw a line segment QR such that QR = CD and QR // CD.

By definition $\overrightarrow{AB} = \overrightarrow{PQ} = \underline{a}$ and $\overrightarrow{CD} = \overrightarrow{QR} = \underline{b}$

According to the triangle of law of addition

$$\overrightarrow{PR} = \overrightarrow{PQ} + \overrightarrow{QR} = \underline{a} + \underline{b}$$



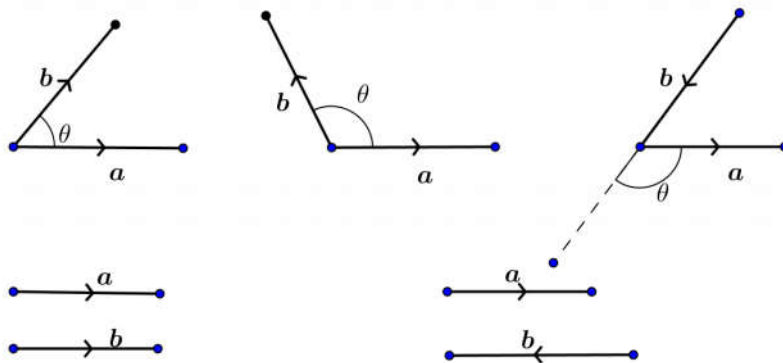
1.12 Definition of a vector

A vector has magnitude and direction and obeys the triangle of law of addition.

1.13 Angle between two vectors

Let \underline{a} and \underline{b} be two vectors.

The angle θ between \underline{a} and \underline{b} is shown below.



Note that $0 \leq \theta \leq \pi$

If \underline{a} and \underline{b} are parallel and are in the same direction, then $\theta = 0$.

If \underline{a} and \underline{b} are parallel and are in the opposite direction, then $\theta = \pi$.

1.14 Position vector

With a fixed point O chosen as the origin, the position of any point P can be denoted by the vector \overrightarrow{OP} .

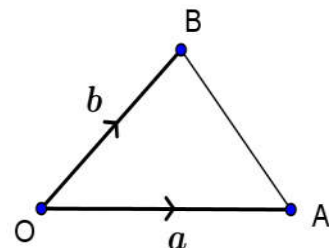
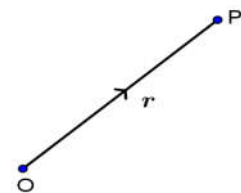
The vector $\overrightarrow{OP} = \underline{r}$ is (known as the) position vector of P with respect to O.

Let the position vectors of two points A and B be \underline{a} and \underline{b} .

$$\overrightarrow{OA} = \underline{a}, \overrightarrow{OB} = \underline{b}$$

$$\overrightarrow{OA} + \overrightarrow{AB} = \overrightarrow{OB}$$

$$\begin{aligned} \overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\ &= \underline{b} - \underline{a} \end{aligned}$$



1.15 Laws of vector algebra

Let $\underline{a}, \underline{b}, \underline{c}$ be vectors and λ, μ be scalars.

- (i) $\underline{a} + \underline{b} = \underline{b} + \underline{a}$ (Commutative Law)
- (ii) $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$ (Associative Law)
- (iii) $\lambda(\underline{a} + \underline{b}) = \lambda\underline{a} + \lambda\underline{b}$ (Distributive Law)
- (iv) $\underline{a} + \underline{0} = \underline{a} = \underline{0} + \underline{a}$
- (v) $\underline{a} + (-\underline{a}) = \underline{0} = (-\underline{a}) + \underline{a}$
- (vi) $(\lambda + \mu)\underline{a} = \lambda\underline{a} + \mu\underline{a}$
- (vii) $\lambda\mu(\underline{a}) = \lambda[\mu\underline{a}] = \mu[\lambda\underline{a}]$

proof:

- (i) Let $\vec{AB} = \underline{a}$ and $\vec{BC} = \underline{b}$
Complete the parallelogram ABCD

Now $\vec{DC} = \vec{AB} = \underline{a}$

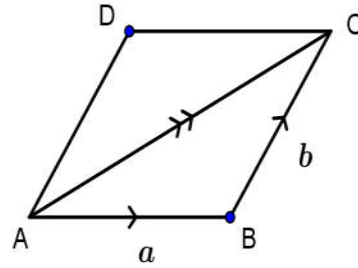
$\vec{AD} = \vec{BC} = \underline{b}$

By triangle law of vector addition

$\vec{AC} = \vec{AB} + \vec{BC} = \underline{a} + \underline{b}$

$\vec{AC} = \vec{AD} + \vec{DC} = \underline{b} + \underline{a}$

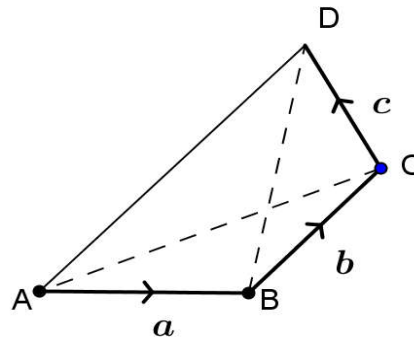
Hence $\underline{a} + \underline{b} = \underline{b} + \underline{a}$



- (ii) Let $\vec{AB} = \underline{a}$, $\vec{BC} = \underline{b}$ and $\vec{CD} = \underline{c}$

$\vec{AD} = \vec{AB} + \vec{BD}$
 $= \vec{AB} + (\vec{BC} + \vec{CD})$
 $= \underline{a} + (\underline{b} + \underline{c}) \dots\dots\dots \textcircled{1}$

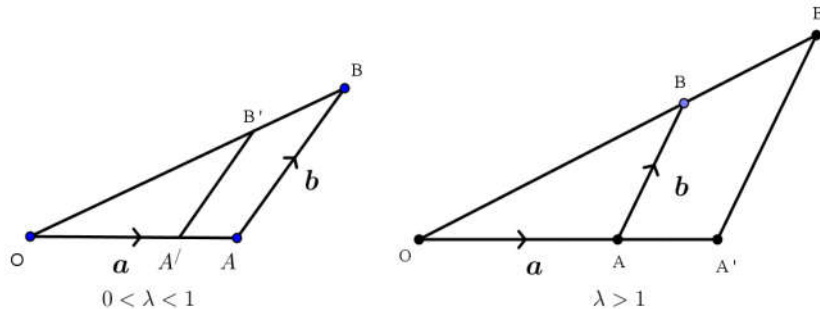
$\vec{AD} = \vec{AC} + \vec{CD}$
 $= (\vec{AB} + \vec{BC}) + \vec{CD}$
 $= (\underline{a} + \underline{b}) + \underline{c} \dots\dots\dots \textcircled{2}$



From (1) and (2) $(\underline{a} + \underline{b}) + \underline{c} = \underline{a} + (\underline{b} + \underline{c})$

- (iii) $\lambda(\underline{a} + \underline{b}) = \lambda\underline{a} + \lambda\underline{b}$

Let $\vec{OA} = \underline{a}$ and $\vec{OB} = \underline{b}$



Take the point A' on OA (or OA produced)

such that $\vec{OA'} = \lambda \vec{OA} = \lambda \underline{a}$

The line drawn parallel to AB through A' meet OB (or OB produced) at B'.

Now, $\Delta OAB, \Delta OA'B'$ are similar triangles.

$$\frac{OA'}{OA} = \frac{A'B'}{AB} = \frac{OB'}{OB} = \lambda.$$

$$\overrightarrow{A'B'} = \lambda \overrightarrow{AB} = \lambda \underline{\mathbf{b}} \text{ and } \overrightarrow{OB'} = \lambda \overrightarrow{OB} \dots\dots\dots \textcircled{1}$$

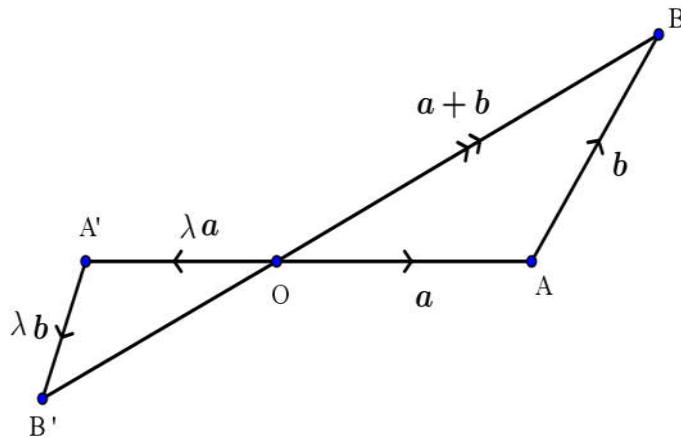
$$\overrightarrow{OB'} = \overrightarrow{OA'} + \overrightarrow{A'B'} = \lambda \underline{\mathbf{a}} + \lambda \underline{\mathbf{b}} \dots\dots\dots \textcircled{2}$$

$$\lambda \overrightarrow{OB} = \lambda(\overrightarrow{OA} + \overrightarrow{AB}) = \lambda(\underline{\mathbf{a}} + \underline{\mathbf{b}}) \dots\dots\dots \textcircled{3}$$

From (i) $\overrightarrow{OB'} = \lambda \overrightarrow{OB}$

$$\lambda \underline{\mathbf{a}} + \lambda \underline{\mathbf{b}} = \lambda(\underline{\mathbf{a}} + \underline{\mathbf{b}})$$

When $\lambda < 0$



$$\overrightarrow{OA} = \underline{\mathbf{a}}, \overrightarrow{AB} = \underline{\mathbf{b}}, \overrightarrow{OA'} = \lambda \underline{\mathbf{a}}$$

$A'B'$ is drawn parallel to BA and meets BO produced at B' . $\overrightarrow{A'B'} = \lambda \underline{\mathbf{b}}$ and $\overrightarrow{OB'} = \lambda \underline{\mathbf{a}} + \lambda \underline{\mathbf{b}}$

By the properties of similar triangles and vectors it can be easily proved that

$$\lambda(\underline{\mathbf{a}} + \underline{\mathbf{b}}) = \lambda \underline{\mathbf{a}} + \lambda \underline{\mathbf{b}} \text{ when } \lambda < 0$$

(iv) $\underline{\mathbf{a}} + \underline{\mathbf{0}} = \underline{\mathbf{a}} = \underline{\mathbf{0}} + \underline{\mathbf{a}}$

Let $\overrightarrow{AB} = \underline{\mathbf{a}}$

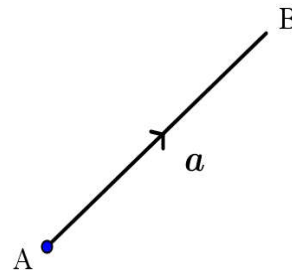
$$\overrightarrow{AB} = \overrightarrow{AB} + \overrightarrow{BB}$$

$$\underline{\mathbf{a}} = \underline{\mathbf{a}} + \underline{\mathbf{0}} \dots\dots\dots \textcircled{1}$$

$$\overrightarrow{AB} = \overrightarrow{AA} + \overrightarrow{AB}$$

$$\underline{\mathbf{a}} = \underline{\mathbf{0}} + \underline{\mathbf{a}} \dots\dots\dots \textcircled{2}$$

From (1) and (2) $\underline{\mathbf{a}} + \underline{\mathbf{0}} = \underline{\mathbf{a}} = \underline{\mathbf{0}} + \underline{\mathbf{a}}$



1.16 Worked examples

Example 1

ABCDEF is a regular hexagon. If $\overrightarrow{AB} = \underline{a}$ and $\overrightarrow{BC} = \underline{b}$, express the vectors \overrightarrow{AC} ,

\overrightarrow{AD} , \overrightarrow{AE} and \overrightarrow{AF} in terms of \underline{a} , \underline{b}

$$\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{BC} = \underline{a} + \underline{b} \quad \dots\dots\dots ①$$

By geometry $AD = 2BC$; $AD \parallel BC$

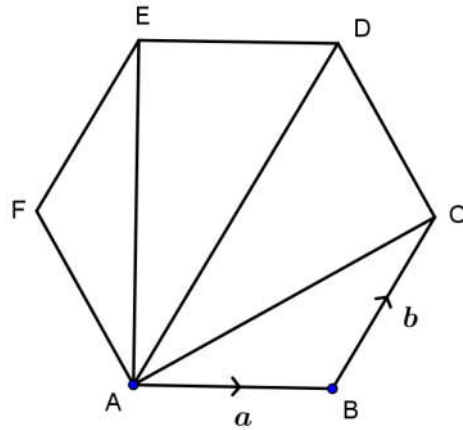
$$\therefore \overrightarrow{AD} = 2\overrightarrow{BC} = 2\underline{b} \quad \dots\dots\dots ②$$

$$\begin{aligned} \overrightarrow{AE} &= \overrightarrow{AD} + \overrightarrow{DE} \\ &= 2\underline{b} + (-\underline{a}) = 2\underline{b} - \underline{a} \quad \dots\dots\dots ③ \end{aligned}$$

By geometry $BC = FE$; $BC \parallel FE$

$$\overrightarrow{FE} = \overrightarrow{BC} = \underline{b}$$

$$\overrightarrow{AF} = \overrightarrow{AE} + \overrightarrow{EF} = (2\underline{b} - \underline{a}) - \underline{b} = \underline{b} - \underline{a}$$



Example 2

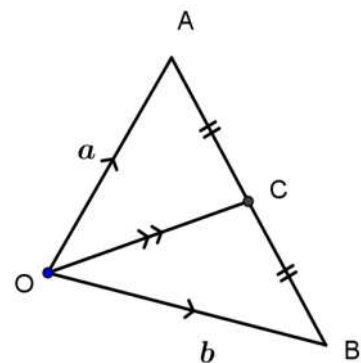
The position vectors of A and B are \underline{a} and \underline{b} respectively

- (i) C is the midpoint of AB.
- (ii) D is a point on AB such that $AD : DB = 1 : 2$
- (iii) E is a point on AB such that $AE : EB = 2 : 1$

Find the position vectors of C, D and E

Let $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$. Then $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \underline{b} - \underline{a}$

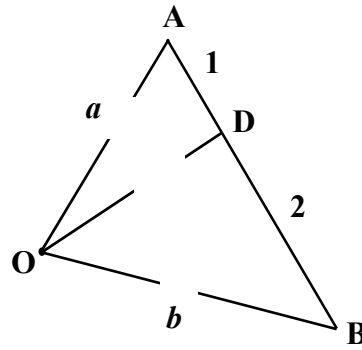
$$\begin{aligned} \text{(i) } AC &= CB \\ \overrightarrow{OC} &= \overrightarrow{OA} + \overrightarrow{AC} \\ &= \overrightarrow{OA} + \frac{1}{2} \overrightarrow{AB} \\ &= \underline{a} + \frac{1}{2} (\underline{b} - \underline{a}) \\ &= \frac{1}{2} (\underline{a} + \underline{b}) \end{aligned}$$



(ii)

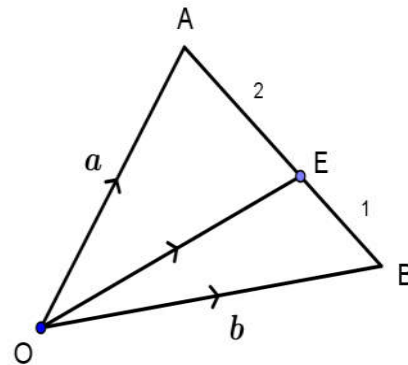
$$AD : DB = 1 : 2$$

$$\begin{aligned}
\overrightarrow{OD} &= \overrightarrow{OA} + \overrightarrow{AD} \\
&= \overrightarrow{OA} + \frac{1}{3}\overrightarrow{AB} \\
&= \overrightarrow{OA} + \frac{1}{3}(\overrightarrow{OB} - \overrightarrow{OA}) \\
&= \underline{a} + \frac{1}{3}(\underline{b} - \underline{a}) \\
&= \frac{2}{3}\underline{a} + \frac{1}{3}\underline{b} = \frac{1}{3}(2\underline{a} + \underline{b})
\end{aligned}$$



(iii)

$$\begin{aligned}
\overrightarrow{OE} &= \overrightarrow{OA} + \overrightarrow{AE} \\
&= \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB} \\
&= \overrightarrow{OA} + \frac{2}{3}(\overrightarrow{OB} - \overrightarrow{OA}) \\
&= \underline{a} + \frac{2}{3}(\underline{b} - \underline{a}) \\
&= \frac{1}{3}\underline{a} + \frac{2}{3}\underline{b} \\
&= \frac{1}{3}(\underline{a} + 2\underline{b})
\end{aligned}$$

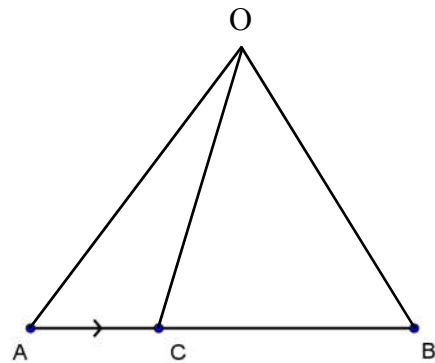


Example 3

Let $-2\mathbf{p} + 5\mathbf{q}$, $7\mathbf{p} - \mathbf{q}$ and $\mathbf{p} + 3\mathbf{q}$ be the position vectors of three points A, B and C respectively, with respect to a fixed origin O, where \mathbf{p} and \mathbf{q} are two non-parallel vectors. Show that the points A, B and C are collinear and find the ratio in which C divides AB.

$$\overrightarrow{OA} = -2\mathbf{p} + 5\mathbf{q}, \quad \overrightarrow{OB} = 7\mathbf{p} - \mathbf{q}, \quad \overrightarrow{OC} = \mathbf{p} + 3\mathbf{q}$$

$$\begin{aligned}
\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} \\
&= (7\mathbf{p} - \mathbf{q}) - (-2\mathbf{p} + 5\mathbf{q}) \\
&= 9\mathbf{p} - 6\mathbf{q} \\
\overrightarrow{AC} &= \overrightarrow{OC} - \overrightarrow{OA} \\
&= (\mathbf{p} + 3\mathbf{q}) - (-2\mathbf{p} + 5\mathbf{q}) \\
&= 3\mathbf{p} - 2\mathbf{q} \\
\overrightarrow{AB} &= 3(3\mathbf{p} - 2\mathbf{q}) \\
\overrightarrow{AC} &= 3\mathbf{p} - 2\mathbf{q} \Rightarrow \overrightarrow{AB} = 3\overrightarrow{AC}
\end{aligned}$$



Therefore A, B and C are collinear and $AC : CB = 1 : 2$

Example 4

\underline{a} , \underline{b} are two non-zero and non parallel vectors and α , β are scalars. Prove that $\alpha\underline{a} + \beta\underline{b} = \underline{0}$ if and only if $\alpha = 0$ and $\beta = 0$.

Assume that $\alpha = 0$ and $\beta = 0$.

$$\alpha\underline{a} + \beta\underline{b} = \underline{0} + \underline{0} = \underline{0}.$$

Conversely let $\alpha\underline{a} + \beta\underline{b} = \underline{0}$

case (i) : Suppose that $\alpha=0$

$$\text{Then } \underline{0} + \beta\underline{b} = \underline{0}$$

$$\beta\underline{b} = \underline{0}$$

since $\underline{b} \neq \underline{0}$, it follows that $\beta = 0$

If $\alpha = 0$, then $\beta = 0$

Similarly we can show that if $\beta = 0$, then $\alpha = 0$

case (ii) : Suppose that $\alpha \neq 0$

$$\alpha\underline{a} + \beta\underline{b} = \underline{0}$$

$$\alpha\underline{a} = -\beta\underline{b}$$

$$\underline{a} = -\frac{\beta}{\alpha}\underline{b} \quad (\alpha \neq \underline{0})$$

The above equation implies that $\underline{a} // \underline{b}$

This is a contradiction

Hence $\alpha=0$ and from the first part $\beta = 0$

ie. $\alpha\underline{a} + \beta\underline{b} = \underline{0}$ if and only if $\alpha = 0, \beta = 0$

Example 5

OABC is a parallelogram. D is the midpoint of BC. OD and AC intersect at M. Given that

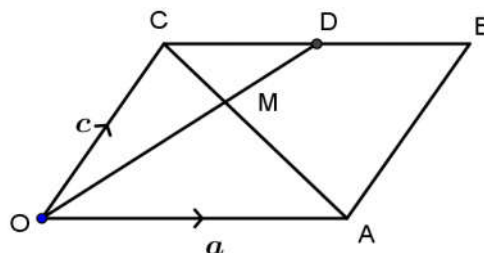
$$\overrightarrow{OA} = \underline{a}, \overrightarrow{OC} = \underline{c}$$

(i) Find \overrightarrow{OD} in terms of \underline{a} and \underline{c}

(ii) If $OM : MD = \lambda : 1$, find \overrightarrow{OM} in terms of \underline{a} , \underline{c} and λ

(iii) If $AM : MC = \mu : 1$, find \overrightarrow{AM} in terms of \underline{a} , \underline{c} and μ and hence find \overrightarrow{OM}

(iv) Using the results obtained in (ii) and (iii) above find the values of λ and μ .



$$(i) \quad \overrightarrow{OA} = \underline{a}, \quad \overrightarrow{OC} = \underline{c}; \quad \overrightarrow{OA} = \overrightarrow{CB} = \underline{a}$$

$$\overrightarrow{OD} = \overrightarrow{OC} + \overrightarrow{CD}$$

$$= \overrightarrow{OC} + \frac{1}{2}\overrightarrow{CB}$$

$$= \underline{c} + \frac{1}{2}\underline{a} \quad \dots\dots\dots ①$$

$$OM : MD = \lambda : 1, \quad \overrightarrow{OM} = \frac{\lambda}{\lambda+1} \overrightarrow{OD} = \frac{\lambda}{\lambda+1} \left[\frac{1}{2}\underline{a} + \underline{c} \right] \quad \dots\dots\dots ②$$

$$\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA}$$

$$= \underline{c} - \underline{a}$$

$$AM : MC = \mu : 1, \quad \overrightarrow{AM} = \frac{\mu}{\mu+1} \overrightarrow{AC} = \frac{\mu}{\mu+1} (\underline{c} - \underline{a})$$

$$\overrightarrow{OM} = \overrightarrow{OA} + \overrightarrow{AM} = \underline{a} + \frac{\mu}{\mu+1} (\underline{c} - \underline{a})$$

$$= \left(1 - \frac{\mu}{\mu+1} \right) \underline{a} + \frac{\mu}{\mu+1} \underline{c}$$

$$= \frac{1}{\mu+1} \underline{a} + \frac{\mu}{\mu+1} \underline{c} \quad \dots\dots\dots ③$$

From ② and ③

Since \underline{a} is not parallel to \underline{c}

$$\frac{\lambda}{2(\lambda+1)} = \frac{1}{\mu+1} \quad \dots\dots\dots ④$$

$$\frac{\lambda}{\lambda+1} = \frac{\mu}{\mu+1} \quad \dots\dots\dots ⑤$$

$$\frac{④}{⑤} \quad \text{gives} \quad \frac{1}{2} = \frac{1}{\mu}, \quad \mu = 2$$

If $\mu = 2$, from ④ $\frac{\lambda}{2(\lambda+1)} = \frac{1}{3}$

$$\lambda = 2$$

$$\lambda = 2 = \mu$$

ie. $OM : MD = AM : MC = 2 : 1$

1.17 Exercises

1. ABCDEF is a regular hexagon. $\overrightarrow{AB} = \underline{a}$, $\overrightarrow{AC} = \underline{b}$. Find \overrightarrow{AD} , \overrightarrow{AE} , \overrightarrow{AF} in terms of \underline{a} , \underline{b} .
2. ABCDEF is a regular hexagon and O is its centre. If $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$ find \overrightarrow{AB} , \overrightarrow{BC} , \overrightarrow{CD} , \overrightarrow{DE} , \overrightarrow{EF} , \overrightarrow{FA} in terms of \underline{a} , \underline{b} .
3. ABCD is a plane quadrilateral and O is a point in the plane of the quadrilateral. If $\overrightarrow{AO} + \overrightarrow{CO} = \overrightarrow{DO} + \overrightarrow{BO}$, show that ABCD is a parallelogram.
4. ABC is an isosceles triangle with $BA = BC$ and D is the midpoint of AC. Show that $\overrightarrow{BA} + \overrightarrow{BC} = 2\overrightarrow{BD}$.
5. \underline{a} and \underline{b} are two vectors perpendicular to each other. Using triangle of law of vector addition show that $|\underline{a} + \underline{b}| = |\underline{a} - \underline{b}|$. When $|\underline{a} - \underline{b}| = 5$ and $|\underline{a}| = 3$, find $|\underline{b}|$.
6. \underline{a} , \underline{b} are two vectors. such that $|\underline{a}| = 6$, $|\underline{b}| = 6$ and the angle between \underline{a} and \underline{b} is 60° . Find $|\underline{a} + \underline{b}|$ and $|\underline{a} - \underline{b}|$.

Use vectors to prove the following questions (7,8,9).

7. ABC is a triangle. D and E are the midpoints of AB and AC. Prove that $DE = \frac{1}{2} BC$ and DE is parallel to BC.
8. ABCD is a quadrilateral. P, Q, R and S are the midpoints of AB, BC, CD and DA respectively. Show that PQRS is a parallelogram.
9. ABC is a triangle. The position vectors of A, B and C are \underline{a} , \underline{b} , \underline{c} respectively. Find the position vector of the centroid of the triangle ABC.
10. OABC is a parallelogram. D is the midpoint of AB. OD and AC intersect at E. $\overrightarrow{OA} = \underline{a}$, $\overrightarrow{OB} = \underline{b}$, $OE : ED = \lambda : 1$, $CE : EA = \mu : 1$.
 - i. Find \overrightarrow{OD} in terms of \underline{a} and \underline{b} . Hence write the vector \overrightarrow{OE} in terms of λ , \underline{a} and \underline{b} .
 - ii. Find the vector \overrightarrow{AC} and write the vector \overrightarrow{OE} in terms of μ , \underline{a} and \underline{b} .
 - iii. Using the results obtained in (i) and (ii) above find λ and μ .
 - iv. When OD and CB produced meet at H, find \overrightarrow{OH} .
11. Let OABC be a quadrilateral and let D and E be the midpoints of the diagonal OB and AC respectively. Also let F be the mid-point of DE. By taking the position vectors of the points A, B and C with respect to O be \underline{a} , \underline{b} and \underline{c} respectively, show that $\overrightarrow{OF} = \frac{1}{4}(\underline{a} + \underline{b} + \underline{c})$.
Let P and Q be the midpoints of the sides OA and BC respectively. Show that the points P, F and Q are collinear and find the ratio PF : FQ.

12. Let A and B two distinct points not collinear with a point O. The position vectors of A and B with respect to the point O is \underline{a} and \underline{b} . If D is the point on AB such that $BD = 2DA$. Show that the position vector of D with respect to point O is $\frac{1}{3}(2\underline{a} + \underline{b})$.

If $\overrightarrow{BC} = K\underline{a}$ ($K > 1$) and the points O, D and C are collinear, find the value of K and the ratio $OD : DC$ Express \overrightarrow{AC} in terms of \underline{a} and \underline{b} .

Further if the line through O parallel to AC meets AB at E show that $6DE = AB$.

13. Let ABCD is a trapezium such that $\overrightarrow{DC} = \frac{1}{2}\overrightarrow{AB}$. Also let $\overrightarrow{AB} = \underline{p}$ and $\overrightarrow{AD} = \underline{q}$. The point E lies on BC such that $\overrightarrow{BE} = \frac{1}{3}\overrightarrow{BC}$. The point of intersection F of AE and BD satisfies $\overrightarrow{BF} = \lambda\overrightarrow{BD}$. where λ ($0 < \lambda < 1$) is a constant. Show that $\overrightarrow{AF} = (1-\lambda)\underline{p} + \lambda\underline{q}$. Hence find the value of λ .

1.18 Cartesian vector notation

Consider the cartesian plane xOy .

Let the unit vector in the direction Ox be \underline{i} , the unit vector in the direction Oy be \underline{j} , and $P \equiv (x, y)$

Let $\overrightarrow{OP} = \underline{r}$

$$\underline{r} = \overrightarrow{OP} = \overrightarrow{OM} + \overrightarrow{MP} = x\underline{i} + y\underline{j} \quad (|\overrightarrow{OM}| = OM = x, |\overrightarrow{MP}| = MP = y)$$

$$|\underline{r}| = OP = \sqrt{x^2 + y^2}$$

Let $\underline{a} = a_1\underline{i} + a_2\underline{j}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j}$

$$\begin{aligned} \underline{a} + \underline{b} &= (a_1\underline{i} + b_1\underline{i}) + (a_2\underline{j} + b_2\underline{j}) \\ &= (a_1 + b_1)\underline{i} + (a_2 + b_2)\underline{j} \end{aligned}$$

Proof

$$\overrightarrow{OA} = a_1\underline{i} + a_2\underline{j} \quad A \equiv (a_1, a_2)$$

$$\overrightarrow{OB} = b_1\underline{i} + b_2\underline{j} \quad B \equiv (b_1, b_2)$$

OACB is a parallelogram $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{OB} = \underline{a} + \underline{b}$

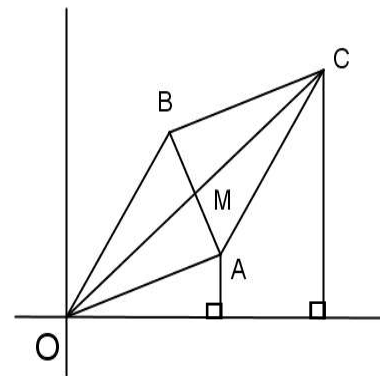
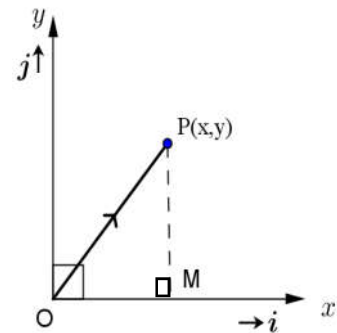
Since M is the midpoint of AB, $M \equiv \left(\frac{a_1 + b_1}{2}, \frac{a_2 + b_2}{2} \right)$

M is the midpoint of OC. $C \equiv (a_1 + b_1, a_2 + b_2)$

$$\overrightarrow{OC} = (a_1 + b_1)\underline{i} + (a_2 + b_2)\underline{j}$$

If $\underline{a} = a_1\underline{i} + a_2\underline{j}$ and $\underline{b} = b_1\underline{i} + b_2\underline{j}$ then

$$\begin{aligned} \underline{a} - \underline{b} &= \underline{a} + (-\underline{b}) = (a_1\underline{i} + a_2\underline{j}) + (-b_1\underline{i} - b_2\underline{j}) \\ \underline{a} - \underline{b} &= \underline{a} + (-\underline{b}) = (a_1\underline{i} + a_2\underline{j}) + (-b_1\underline{i} - b_2\underline{j}) \\ &= (a_1 - b_1)\underline{i} + (a_2 - b_2)\underline{j} \end{aligned}$$



Example 6

If $A \equiv (2, -1)$ and $B \equiv (5, 3)$ find

i. \overrightarrow{OA} , \overrightarrow{OB} , \overrightarrow{AB} in terms of $\underline{i}, \underline{j}$

ii. $|\overrightarrow{OA}|$, $|\overrightarrow{OB}|$, $|\overrightarrow{AB}|$

iii. the unit vector in the direction \overrightarrow{AB}

$A \equiv (2, -1)$, $B \equiv (5, 3)$

(i) $\overrightarrow{OA} = 2\underline{i} - \underline{j}$ $\overrightarrow{OB} = 5\underline{i} + 3\underline{j}$

$$\begin{aligned}\overrightarrow{AB} &= \overrightarrow{OB} - \overrightarrow{OA} = (5\underline{i} + 3\underline{j}) - (2\underline{i} - \underline{j}) \\ &= 3\underline{i} + 4\underline{j}\end{aligned}$$

(ii) $\overrightarrow{OA} = 2\underline{i} - \underline{j}$, $|\overrightarrow{OA}| = \sqrt{2^2 + 1^2} = \sqrt{5}$

$$\overrightarrow{OB} = 5\underline{i} + 3\underline{j}, \quad |\overrightarrow{OB}| = \sqrt{5^2 + 3^2} = \sqrt{34}$$

$$\overrightarrow{AB} = 3\underline{i} + 4\underline{j}, \quad |\overrightarrow{AB}| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

(iii) unit vector in the direction \overrightarrow{AB} is

$$\frac{\overrightarrow{AB}}{|\overrightarrow{AB}|} = \frac{1}{5}(3\underline{i} + 4\underline{j})$$

1.19 Exercises

1. Let $\underline{a} = \underline{i} - 2\underline{j}$ $\underline{b} = 4\underline{j}$ $\underline{c} = 3\underline{i} - \underline{j}$ Find

i. (a) $2\underline{a} + \underline{b}$ (b) $\underline{a} + 3\underline{c}$ (c) $2\underline{a} - \underline{b} - \underline{c}$

ii. (a) $|2\underline{a} + \underline{b}|$ (b) $|\underline{a} + 3\underline{c}|$ (c) $|2\underline{a} - \underline{b} - \underline{c}|$

iii. the unit vector in the direction $\underline{a} + \underline{b} + \underline{c}$

2. Given that $A \equiv (4, 3)$, $B \equiv (6, 6)$ and $C \equiv (0, 1)$

(a) Write the vectors \overrightarrow{OA} , \overrightarrow{OB} , \overrightarrow{OC}

(b) Find \overrightarrow{AB} , \overrightarrow{BC} , \overrightarrow{CA}

(c) Find $|\overrightarrow{AB}|$, $|\overrightarrow{BC}|$, $|\overrightarrow{CA}|$

3. O is the origin and $\overrightarrow{OA} = -\underline{i} + 5\underline{j}$ $\overrightarrow{OB} = 2\underline{i} + 4\underline{j}$ and $\overrightarrow{OC} = 2\underline{j}$. Find \overrightarrow{AB} , \overrightarrow{BC} , \overrightarrow{CA} . Hence show that ABC is an isosceles triangle.

4. If $\overrightarrow{OA} = \underline{i} + 2\underline{j}$, $\overrightarrow{OB} = 3\underline{i} - \underline{j}$ and $\overrightarrow{OC} = -\underline{j} + 5\underline{j}$. Find \overrightarrow{AB} and \overrightarrow{CA} and hence show that the points A, B and C are collinear.
5. The position vectors of the points A and B are \underline{a} and \underline{b} respectively, where $\underline{a} = 2\underline{i} + 3\underline{j}$ and $\underline{b} = \underline{i} + 5\underline{j}$.
- (i) If R is the midpoint of AB, find the position vector of R in terms of $\underline{i}, \underline{j}$.
- (ii) If $\underline{c} = 2\underline{a} - \underline{b}$. Find the unit vector along \underline{c} in terms of $\underline{i}, \underline{j}$.
6. (a) Find in the form $a\underline{i} + b\underline{j}$, a vector of magnitude 10 units in the direction $3\underline{i} - 4\underline{j}$
- (b) A $\equiv (-2, -5)$ and B $\equiv (3, 7)$
- (i) Write \overrightarrow{OA} , \overrightarrow{OB} and hence \overrightarrow{AB}
- (ii) Find in the form $a\underline{i} + b\underline{j}$, a vector of magnitude 65 units in the direction \overrightarrow{AB}

1.20 Scalar product of two vectors

We learnt earlier the rules of vector addition and subtraction. Two types of products have been defined.

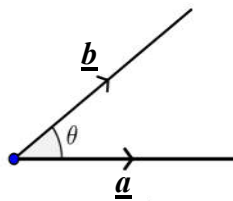
- (i) The scalar product of two vectors.
- (ii) The vector product of two vectors.

The scalar product is also known as the dot product. The result of a dot product is a scalar and the result of the vector product is a vector.

Definition : Scalar Product

Let \underline{a} and \underline{b} be any two non-zero vectors and θ be the angle between the two vectors.

The scalar product of two vectors \underline{a} and \underline{b} is defined as $\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos\theta$ ($0 \leq \theta \leq \pi$)



Properties of the scalar product

1. $\underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$ (Commutative Law)

By definition, $\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos\theta$

$$= |\underline{b}| |\underline{a}| \cos\theta$$

$$\text{Hence } \underline{a} \cdot \underline{b} = \underline{b} \cdot \underline{a}$$

2. If \underline{a} and \underline{b} are two non - zero vectors $\underline{a} \cdot \underline{b} = 0$, if and only if \underline{a} is perpendicular to \underline{b} .

$$\begin{aligned} \underline{a} \cdot \underline{b} = 0 &\Leftrightarrow |\underline{a}| |\underline{b}| \cos\theta = 0 \\ &\Leftrightarrow \cos\theta = 0 \quad (\underline{a}, \underline{b} \neq \underline{0}) \\ &\Leftrightarrow \theta = \frac{\pi}{2} \end{aligned}$$

3. $\underline{a} \cdot \underline{a} = |\underline{a}| |\underline{a}| \cos 0 = |\underline{a}|^2$ also written \underline{a}^2

$$\underline{i} \cdot \underline{i} = |\underline{i}| |\underline{i}| \cos 0 = 1 \times 1 \times 1 = 1$$

$$\underline{j} \cdot \underline{j} = |\underline{j}| |\underline{j}| \cos 0 = 1 \times 1 \times 1 = 1$$

$$\underline{i} \cdot \underline{j} = |\underline{i}| |\underline{j}| \cos \frac{\pi}{2} = 1 \times 1 \times 0 = 0$$

i.e: $\underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = 1$ and $\underline{i} \cdot \underline{j} = \underline{j} \cdot \underline{i} = 0$

4. $\underline{a}, \underline{b}, \underline{c}$ are vectors.

$$\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c} \quad (\text{Distributive Law})$$

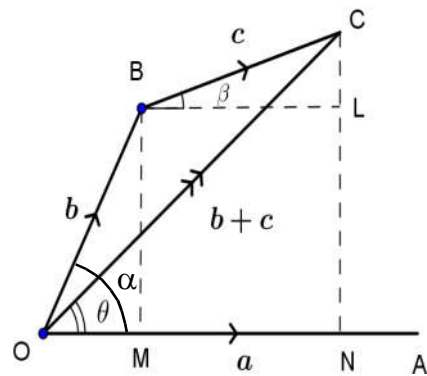
Let the angle between

\underline{a} and \underline{b} be α

\underline{a} and \underline{c} be β

\underline{a} and $(\underline{b} + \underline{c})$ be θ

$$\begin{aligned} \underline{a} \cdot (\underline{b} + \underline{c}) &= |\underline{a}| \cdot |\underline{b} + \underline{c}| \cos\theta \\ &= (\text{OA}) (\text{OC}) \cos\theta \\ &= (\text{OA}) \cdot (\text{ON}) \\ &= (\text{OA}) (\text{OM} + \text{MN}) \\ &= \text{OA} \cdot \text{OM} + \text{OA} \cdot \text{MN} \\ &= \text{OA} \cdot \text{OB} \cos\alpha + \text{OA} \cdot \text{BC} \cos\beta \quad (\text{MN} = \text{BL}) \\ &= \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c} \\ &= \overrightarrow{\text{OA}} \cdot \overrightarrow{\text{OB}} + \overrightarrow{\text{OA}} \cdot \overrightarrow{\text{BC}} \end{aligned}$$



Therefore $\underline{a} \cdot (\underline{b} + \underline{c}) = \underline{a} \cdot \underline{b} + \underline{a} \cdot \underline{c}$

5. Let $\underline{a} = a_1 \underline{i} + a_2 \underline{j}$ and $\underline{b} = b_1 \underline{i} + b_2 \underline{j}$

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (a_1 \underline{i} + a_2 \underline{j}) \cdot (b_1 \underline{i} + b_2 \underline{j}) \\ &= a_1 \underline{i} \cdot (b_1 \underline{i} + b_2 \underline{j}) + a_2 \underline{j} \cdot (b_1 \underline{i} + b_2 \underline{j}) \\ &= a_1 \underline{i} \cdot b_1 \underline{i} + a_1 \underline{i} \cdot b_2 \underline{j} + a_2 \underline{j} \cdot b_1 \underline{i} + a_2 \underline{j} \cdot b_2 \underline{j} \\ &= a_1 b_1 + a_2 b_2 \quad (\text{since } \underline{i} \cdot \underline{i} = \underline{j} \cdot \underline{j} = 1 \text{ and } \underline{i} \cdot \underline{j} = \underline{j} \cdot \underline{i} = 0) \end{aligned}$$

Example 7

$\underline{a} = 2\underline{i} - 3\underline{j}$ and $\underline{b} = \underline{i} - 3\underline{j}$ Find the angle between \underline{a} and \underline{b}

$$\underline{a} \cdot \underline{b} = |\underline{a}| |\underline{b}| \cos\theta$$

$$|\underline{a}| = \sqrt{2^2 + 3^2} = \sqrt{13} \quad |\underline{b}| = \sqrt{1^2 + 3^2} = \sqrt{10}$$

$$\underline{a} \cdot \underline{b} = \sqrt{13} \times \sqrt{10} \cos\theta \quad \text{①}$$

$$\begin{aligned} \underline{a} \cdot \underline{b} &= (2\underline{i} - 3\underline{j}) \cdot (\underline{i} - 3\underline{j}) \\ &= 2\underline{i} \cdot (\underline{i} - 3\underline{j}) - 3\underline{j} \cdot (\underline{i} - 3\underline{j}) \\ &= 2 + 0 - 0 + 9 = 11 \quad \text{②} \end{aligned}$$

From (1) and (2) $\sqrt{130} \cos\theta = 11$

$$\cos\theta = \frac{11}{\sqrt{130}}, \theta = \cos^{-1}\left(\frac{11}{\sqrt{130}}\right)$$

Example 8

i. If $\underline{a}, \underline{b}$ are two vectors such that $|\underline{a}| = |\underline{b}| = |\underline{a} + \underline{b}|$,

Find the angle between \underline{a} and \underline{b} .

ii. Two vectors \underline{a} and \underline{b} are such that \underline{a} is perpendicular to $\underline{a} + \underline{b}$. If $|\underline{b}| = \sqrt{2}|\underline{a}|$,

Show that $(2\underline{a} + \underline{b})$ is perpendicular to \underline{b} .

i. $|\underline{a}| = |\underline{b}| = |\underline{a} + \underline{b}|$

$$|\underline{a}|^2 = |\underline{a} + \underline{b}|^2$$

$$\underline{a} \cdot \underline{a} = (\underline{a} + \underline{b}) \cdot (\underline{a} + \underline{b}) \text{ (by definition)}$$

$$|\underline{a}|^2 = |\underline{a}|^2 + |\underline{b}|^2 + 2\underline{a} \cdot \underline{b}$$

$$- |\underline{b}|^2 = 2|\underline{a}| |\underline{b}| \cos\theta$$

$$- |\underline{b}|^2 = 2|\underline{b}| |\underline{b}| \cos\theta$$

$$\cos\theta = -\frac{1}{2} \quad \theta = \frac{2\pi}{3}$$

ii. $(\underline{a} + \underline{b}) \cdot \underline{a} = 0 \Rightarrow \underline{a} \cdot \underline{a} + \underline{a} \cdot \underline{b} = 0$
 $|\underline{a}|^2 + \underline{a} \cdot \underline{b} = 0 \dots\dots\dots \text{①}$

$$\begin{aligned} (2\underline{a} + \underline{b}) \cdot \underline{b} &= 2\underline{a} \cdot \underline{b} + \underline{b} \cdot \underline{b} \\ &= 2\underline{a} \cdot \underline{b} + |\underline{b}|^2 \\ &= -2|\underline{a}|^2 + |\underline{b}|^2 \quad \text{(from ①)} \\ &= -2|\underline{a}|^2 + 2|\underline{a}|^2 \quad \text{(since } |\underline{b}| = \sqrt{2}\underline{a} \text{)} \\ &= 0 \end{aligned}$$

Hence $(2\underline{a} + \underline{b})$ is perpendicular to \underline{b} .

Example 9

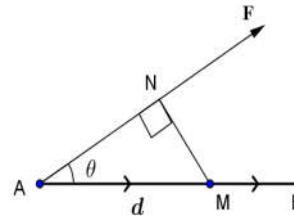
If a constant force \underline{F} acting on a body moves it a distance d in the direction \overline{AB} , where \overline{AB} makes an angle θ with \underline{F} , the work done by the force \underline{F} is $\underline{F} \cdot \underline{d}$.

If the point of application of a force $\underline{F} = 2\underline{i} + 3\underline{j}$ makes a displacement $\underline{S} = 5\underline{i} - 3\underline{j}$,

Find the workdone by the force \underline{F} .

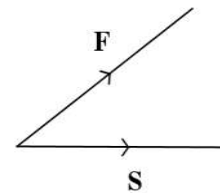
Workdone by \underline{F} is

$$\begin{aligned} &= |\underline{F}| \cdot AN \\ &= |\underline{F}| \cdot AM \cos\theta \quad (\overline{AM} = d) \\ &= \underline{F} \cdot \underline{d} \end{aligned}$$



Workdone by $\underline{F} \cdot \underline{S}$

$$\begin{aligned} &= (2\underline{i} + 3\underline{j}) \cdot (5\underline{i} - 3\underline{j}) \\ &= 2 \times 5 - 3 \times 3 = 10 - 9 = 1 \text{ Joule} \end{aligned}$$



1.21 Exercises

1. If $\underline{a} = 3\underline{i} + \underline{j}$ and $\underline{b} = -\underline{i} + 2\underline{j}$, Find the angle between \underline{a} and \underline{b} .
2. If $\underline{a} = p\underline{i} + 3\underline{j}$ and $\underline{b} = 2\underline{i} + 6\underline{j}$ are two perpendicular vectors,
 - i. find the value of p
 - ii. find $|\underline{a}|$ and $|3\underline{b} - \underline{a}|$
 - iii. find $\underline{a} \cdot (3\underline{b} - \underline{a})$
 - iv. find the angle between \underline{a} and $(3\underline{b} - \underline{a})$
3. Two vectors \underline{a} and \underline{b} are such that $|\underline{a}| = |\underline{b}| = |\underline{a} - \underline{b}|$.
Find the angle between \underline{a} and \underline{b}
4. If $|\underline{a}| = 3$, $|\underline{b}| = 2$ and $|\underline{a} - \underline{b}| = 4$,
Find (i) $\underline{a} \cdot \underline{b}$
(ii) $|\underline{a} + \underline{b}|$
5. If \underline{a} and $(\underline{a} + \underline{b})$ are perpendicular vectors to each other,
Show that $|\underline{a} + \underline{b}|^2 = |\underline{b}|^2 - |\underline{a}|^2$.
6. Using the dot product, show that the diagonals of a rhombus are perpendicular to each other.
7. Show that if $|\underline{a} + \underline{b}| = |\underline{a} - \underline{b}|$ then $\underline{a} \cdot \underline{b} = 0$. Hence, show that if the diagonals of a parallelogram are equal, then it is a rectangle.
8. $\underline{a} = \underline{i} + \sqrt{3}\underline{j}$ where \underline{i} and \underline{j} have the usual meaning. \underline{b} is a vector with magnitude $\sqrt{3}$.
If the angle between the vectors \underline{a} and \underline{b} is $\frac{\pi}{3}$, Find \underline{b} in the form $x\underline{i} + y\underline{j}$ where $x (<0)$ and y are constants to be determined.
9. AB is a diameter of a circle and P is any point on the circumference of the circle. show that APB is a right angle (use dot product)
10. Using dot product, with the usual notation prove that for any triangle ABC,
$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

2.0 System of coplanar forces acting on a particle

2.1 Introduction

Statics :

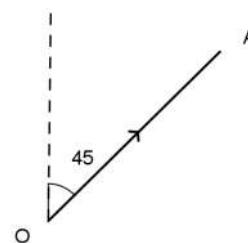
Statics is a branch of mechanics. It deals with bodies in equilibrium under the action of forces.

Force :

Force is defined as any cause which alters or tend to alter a body's state of rest or of uniform motion in a straight line. Unit of force is Newton and denoted by N.

To specify a force which acts on a particle it is important to give.

- i. magnitude of the force
- ii. direction of the force and
- iii. point of its application



Force can be represented by a directed line segment. Let a force 10 Newton (N) is acting at a point O in the north - east direction. Then the force can be represented by the directed line segment OA. In this diagram the length OA represents 10 units and the arrow mark gives its direction.

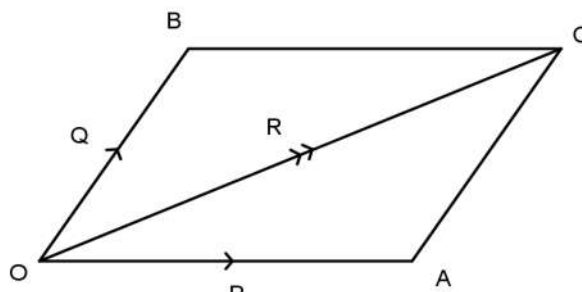
Resultant force:

When a body is acted upon by a number of forces, a single force equivalent to the given forces is called resultant force.

2.2 The parallelogram law of forces

The parallelogram law of forces is the fundamental theorem of Statics and it can be verified by experiment.

If two forces, acting on a particle at O, be represented in magnitude and direction by the two straight lines OA and OB respectively, then the resultant is represented in magnitude and direction by the diagonal OC of the parallelogram OACB.



Let the forces P and Q be represented by OA and OB respectively. Then the resultant R of P and Q is represented by the diagonal OC of the parallelogram OACB.

Let the angle between P and Q be θ and the resultant R makes an angle α with P.

By Pythagora's theorem

$$OC^2 = OM^2 + MC^2$$

$$= (OA + AM)^2 + MC^2$$

$$R^2 = (P + Q \cos\theta)^2 + (Q \sin\theta)^2$$

$$= P^2 + 2PQ \cos\theta + Q^2 \cos^2\theta + Q^2 \sin^2\theta$$

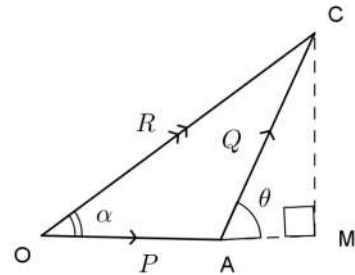
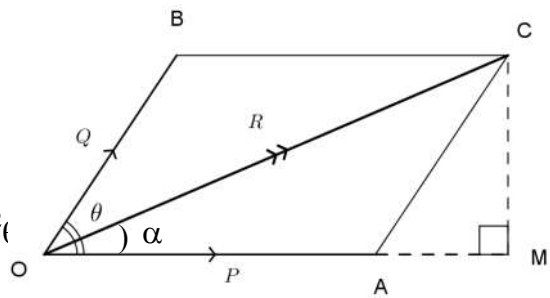
$$R^2 = P^2 + Q^2 + 2PQ \cos\theta$$

$$\tan \alpha = \frac{CM}{OM} = \frac{CM}{OA+AM}$$

$$= \frac{Q \sin\theta}{P+Q \cos\theta}$$

$$R^2 = P^2 + Q^2 + 2PQ \cos\theta$$

$$\tan \alpha = \frac{Q \sin\theta}{P+Q \cos\theta}$$



When $\theta = 90^\circ$, $\cos\theta = \cos 90 = 0$; $\sin\theta = \sin 90 = 1$

$$R^2 = P^2 + Q^2, \text{ and } \tan \alpha = \frac{Q}{P}$$

when $Q = P$

$$R^2 = P^2 + P^2 + 2P \times P \times \cos\theta$$

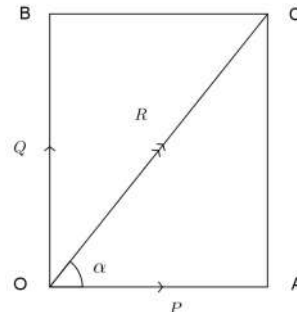
$$= 2P^2 + 2P^2 \cos\theta = 2P^2(1 + \cos\theta)$$

$$= 2P^2 \times 2\cos^2 \frac{\theta}{2} = 4P^2 \cos^2 \frac{\theta}{2}$$

$$R = 2P \cos \frac{\theta}{2}$$

$$\tan \alpha = \frac{P \sin\theta}{P + P \cos\theta} = \frac{\sin\theta}{1 + \cos\theta} = \frac{2\sin \frac{\theta}{2} \cdot \cos \frac{\theta}{2}}{2\cos^2 \frac{\theta}{2}} = \tan \frac{\theta}{2}$$

$$\alpha = \frac{\theta}{2}$$



When the two forces are equal the resultant of these two forces bisects the angle between the forces.

Alternate Method (Using geometry)

When $P = Q$; $OA = OB$

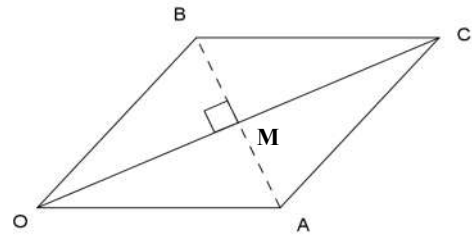
The parallelogram $OACB$ is rhombus.

i. OC and AB intersect at right angles.

$$\text{ii. } \angle AOC = \angle BOC \quad \left(= \frac{\theta}{2} \right)$$

$$OC = 2OM = 2OA \cos \frac{\theta}{2}$$

$$R = 2P \cos \frac{\theta}{2}$$



Example 1:

Forces $3P$ and $5P$ act at a point and the angle between the forces is 60° . Find the resultant.

$$R^2 = P^2 + Q^2 + 2PQ \cos \theta$$

$$= (3P)^2 + (5P)^2 + 2 \times 3P \times 5P \cdot \cos 60^\circ$$

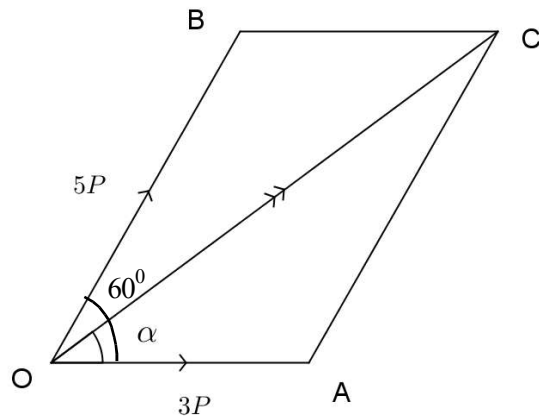
$$= 9P^2 + 25P^2 + 15P^2 = 49P^2$$

$$R = 7P$$

$$\tan \alpha = \frac{5P \sin 60^\circ}{3P + 5P \cos 60^\circ}$$

$$\tan \alpha = \frac{5\sqrt{3}}{11}$$

$$\alpha = \tan^{-1} \left(\frac{5\sqrt{3}}{11} \right)$$



Example 2

The resultant of two forces $8P$ and $5P$ acting at a point is $7P$. Find the angle between these two forces.

Let θ be the angle between the forces $8P$ and $5P$.

$$R^2 = P^2 + Q^2 + 2PQ \cos \theta$$

$$(7P)^2 = (8P)^2 + (5P)^2 + 2 \times 8P \times 5P \cos \theta$$

$$49P^2 = 64P^2 + 25P^2 + 80P^2 \cos \theta$$

$$-40 = 80 \cos \theta$$

$$\cos \theta = -\frac{1}{2}$$

$$\theta = 120^\circ$$

Example 3 :

The resultant of two forces P and $\sqrt{2}P$, acting at a point is at right angle to the smaller force. Find the resultant and the angle between the two given forces.

Applying Pythagora's theorem,

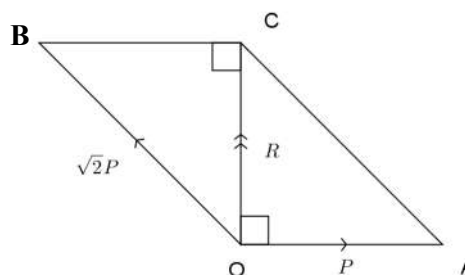
$$OC^2 + CB^2 = OB^2$$

$$R^2 + P^2 = (\sqrt{2}P)^2$$

$$R^2 = P^2; R = P;$$

Therefore $OC = BC$ and $\angle BOC = 45^\circ$

The angle between the force is $90^\circ + 45^\circ = 135^\circ$



2.3 Resolution of a force into two directions

a. Rectangular resolution of a force

We have studied that two forces acting at a point can be reduced to an equivalent single force (resultant) using parallelogram of forces. Conversely a single force can be resolved into pair of component in an infinite number of ways.

Let R be a force acting on a particle. It can be resolved in two perpendicular directions.

Let the force R be represented by OC .

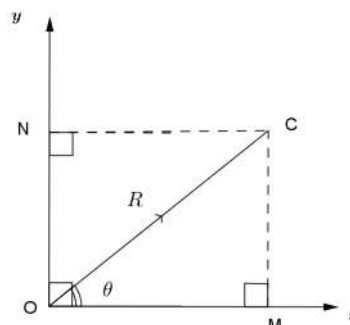
We have to resolve the force R , along Ox and Oy .

Let θ be the angle R makes with Ox axis.

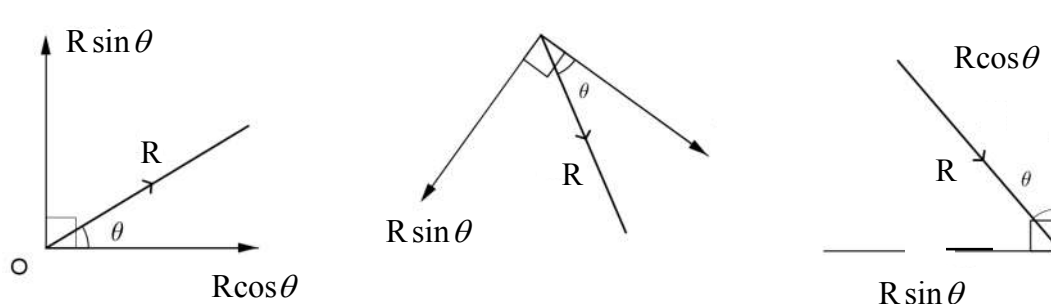
$OMCN$ is a rectangle.

$$\cos\theta = \frac{OM}{OC}, \quad OM = OC \cos\theta = R \cos\theta$$

$$\sin\theta = \frac{MC}{OC}, \quad MC = OC \sin\theta = R \sin\theta = ON$$



Hence the resolved components of R along Ox and Oy are $R \cos\theta$ and $R \sin\theta$ respectively.



b. Oblique Resolution

Let R be the given force and let OA and OB be the given directions along which the force R is to be resolved.

R is represented by OC .

Through C , draw lines CM and CL parallel to OA and OB respectively. Now, $OLCM$ is a parallelogram

Hence OL and OM are the resolved components of R along OA and OB respectively.

Let $\hat{C}OA = \alpha$ and $\hat{C}OB = \beta$

Using sine law in the triangle OLC .

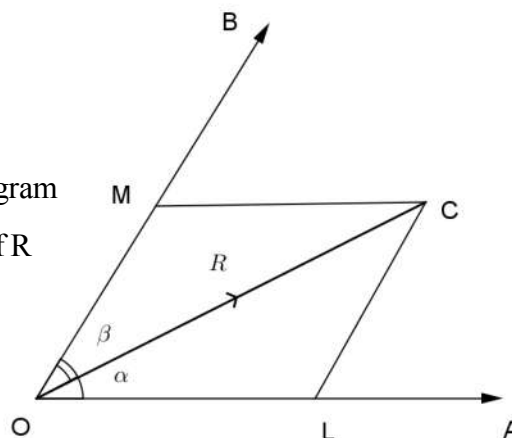
$$\frac{OL}{\sin \hat{O}CL} = \frac{LC}{\sin \hat{C}OL} = \frac{OC}{\sin \hat{O}LC}$$

$$\frac{OL}{\sin \beta} = \frac{LC}{\sin \alpha} = \frac{R}{\sin [180 - (\alpha + \beta)]}$$

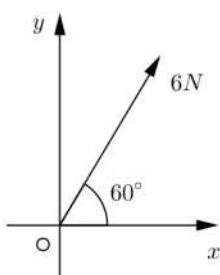
$$\frac{OL}{\sin \beta} = \frac{LC}{\sin \alpha} = \frac{R}{\sin(\alpha + \beta)}$$

$$OL = \frac{R \sin \beta}{\sin(\alpha + \beta)}, \quad LC = \frac{R \sin \alpha}{\sin(\alpha + \beta)}$$

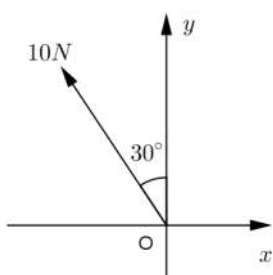
Hence the resolved components along OA, OB are $\frac{R \sin \beta}{\sin(\alpha + \beta)}, \frac{R \sin \alpha}{\sin(\alpha + \beta)}$ respectively.



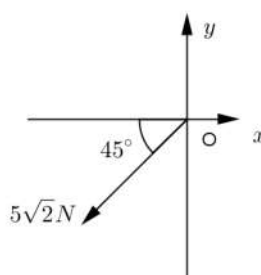
Example 4



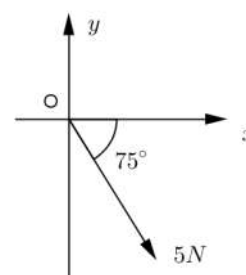
(a)



(b)



(c)



(d)

$$(a) \rightarrow X = 6 \cos 60^\circ = 6 \times \frac{1}{2} = 3N$$

$$\uparrow Y = 6 \sin 60 = 6 \times \frac{\sqrt{3}}{2} = 3\sqrt{3} N$$

$$(b) \quad \leftarrow X = 10\sin 30 = 10 \times \frac{1}{2} = 5\text{N}$$

$$\quad \uparrow Y = 10\cos 30 = 10 \times \frac{\sqrt{3}}{2} = 5\sqrt{3}\text{N}$$

$$(c) \quad \leftarrow X = 5\sqrt{2} \cos 45 = 5\sqrt{2} \times \frac{1}{\sqrt{2}} = 5\text{N}$$

$$\quad \downarrow Y = 5\sqrt{2} \sin 45 = 5\sqrt{2} \times \frac{1}{\sqrt{2}} = 5\text{N}$$

$$(d) \quad \rightarrow X = 5\cos 75 = 5\left(\frac{\sqrt{3}-1}{2\sqrt{2}}\right)\text{N}$$

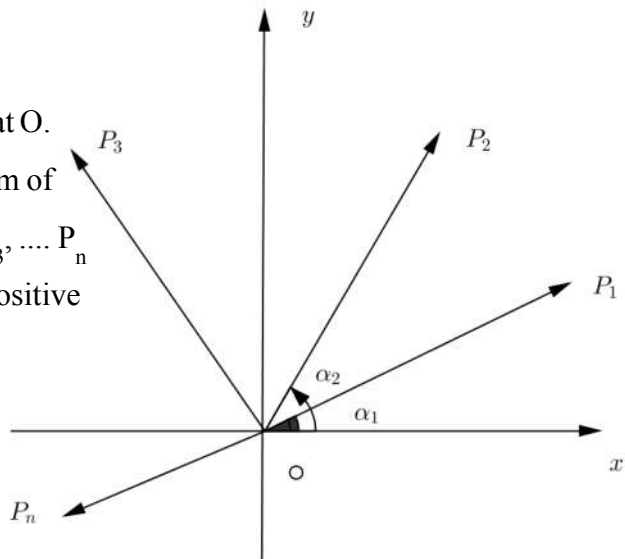
$$\quad \downarrow Y = 5\sin 75 = 5\left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)\text{N}$$

2.4 Resultant of a system of coplanar forces acting at a point

Let Ox and Oy be two perpendicular axes.

In the plane of xoy , a system of forces act at O.

Let $P_1, P_2, P_3, \dots, P_n$ be a coplanar system of forces acting at O and the forces $P_1, P_2, P_3, \dots, P_n$ makes angles $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ with the positive direction of Ox axis.



Resolving along Ox,

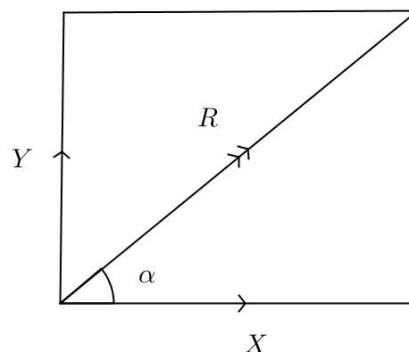
$$\rightarrow X = P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + P_3 \cos \alpha_3 + \dots + P_n \cos \alpha_n$$

$$\uparrow Y = P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + P_3 \sin \alpha_3 + \dots + P_n \sin \alpha_n$$

If R is the resultant, then

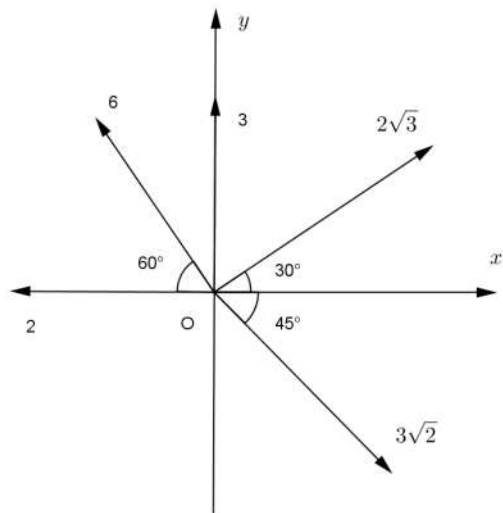
$$R = \sqrt{X^2 + Y^2}$$

$$\tan \alpha = \frac{Y}{X}$$

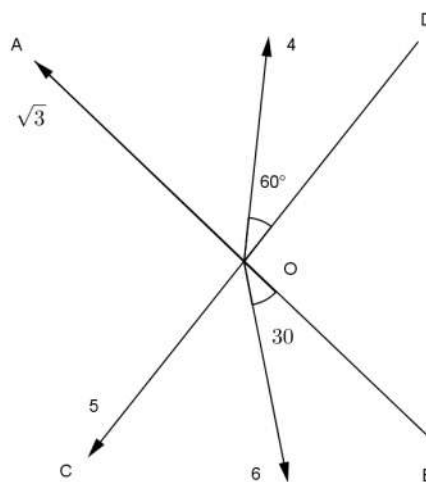


Example 5

For each of the following sets of forces acting at O, find the resultant.



(a)



(b)

Ox and Oy are perpendicular to each other AB and CD are perpendicular to each other

(a) Resolving along Ox,

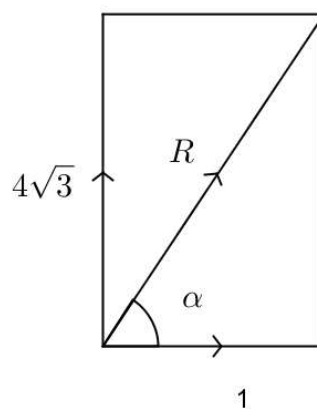
$$\begin{aligned} X &= 2\sqrt{3} \cos 30^\circ - 6 \cos 60^\circ - 2 + 3\sqrt{2} \cos 45^\circ \\ &= 3 - 3 - 2 + 3 = 1 \end{aligned}$$

Resolving along Oy,

$$\begin{aligned} Y &= 3 + 2\sqrt{3} \sin 30^\circ + 6 \sin 60^\circ - 3\sqrt{2} \sin 45^\circ \\ &= 3 + \sqrt{3} + 3\sqrt{3} - 3 = 4\sqrt{3} \end{aligned}$$

$$R^2 = X^2 + Y^2 = (4\sqrt{3})^2 + 1^2 = 49$$

$$R = 7\text{N}, \tan \alpha = 4\sqrt{3}$$



(b) Resolving along BA

$$\begin{aligned} X &= \sqrt{3} + 4 \sin 60^\circ - 6 \cos 30^\circ \\ &= \sqrt{3} + 2\sqrt{3} - 3\sqrt{3} = 0 \end{aligned}$$

Resolving along DC

$$\begin{aligned} Y &= 5 - 4 \cos 60^\circ + 6 \sin 30^\circ \\ &= 5 - 2 + 3 = 6 \end{aligned}$$

Hence the resultant is 6N along DC

Example 6

ABCDEF is a regular hexagon. Forces of magnitudes 2, $4\sqrt{3}$, 8, $2\sqrt{3}$ and 4 newtons act at A in the directions AB, AC, AD, AE and AF respectively. Find the resultant.

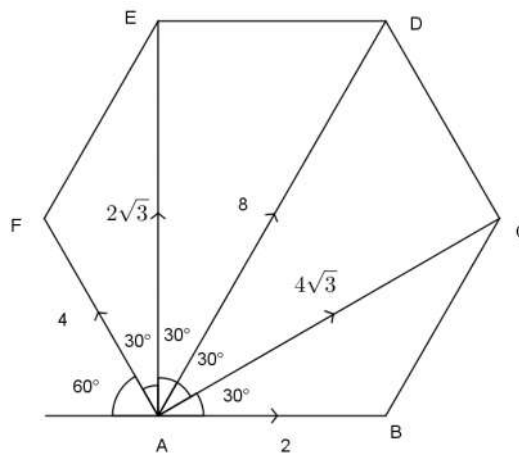
$$\hat{BAE} = 90^\circ$$

Take AB and AE are x and y axis respectively.

Resolving along AB

$$\begin{aligned} X &= 2 + 4\sqrt{3} \cos 30^\circ + 8 \cos 60^\circ - 4 \cos 60^\circ \\ &= 2 + 6 + 4 - 2 = 10 \end{aligned}$$

$$\begin{aligned} Y &= 4\sqrt{3} \sin 30^\circ + 8 \sin 60^\circ + 2\sqrt{3} + 4 \sin 60^\circ \\ &= 2\sqrt{3} + 4\sqrt{3} + 2\sqrt{3} + 2\sqrt{3} = 10\sqrt{3} \end{aligned}$$

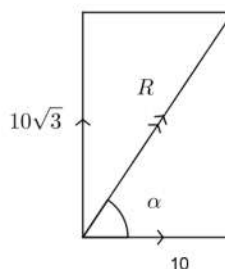


$$R^2 = X^2 + Y^2 = 10^2 + (10\sqrt{3})^2$$

$$= 400$$

$$R = 20\text{N}$$

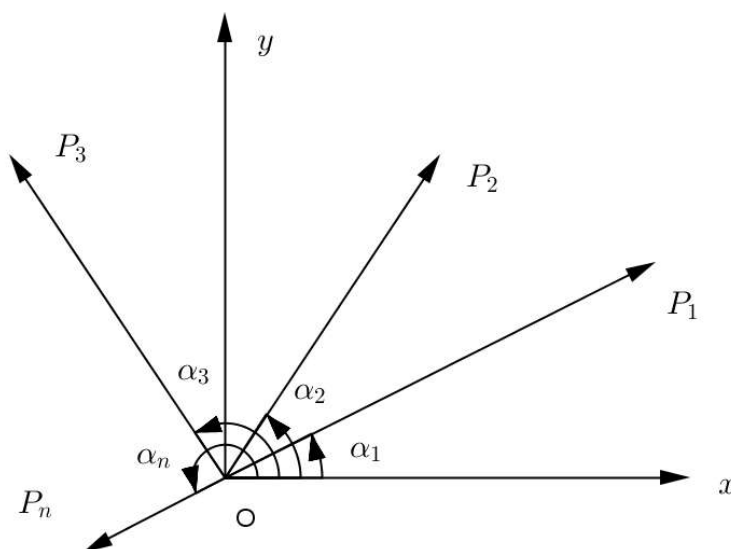
$$\tan \alpha = \frac{10\sqrt{3}}{10} = \sqrt{3}; \quad \alpha = 60^\circ$$



Hence the resultant is 20N and act along AD.

2.5 Equilibrium of coplanar forces acting at a point

Let Ox and Oy be two perpendicular axes. In the plane of xOy, a system of forces act at O.



Let $P_1, P_2, P_3, \dots, P_n$ be a set of coplanar forces acting at O and the forces $P_1, P_2, P_3, \dots, P_n$ make angles $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_n$ with the positive direction of Ox axis.

Resolving along Ox axis

$$\begin{aligned} \rightarrow X &= P_1 \cos \alpha_1 + P_2 \cos \alpha_2 + \dots + P_n \cos \alpha_n \\ \uparrow Y &= P_1 \sin \alpha_1 + P_2 \sin \alpha_2 + \dots + P_n \sin \alpha_n \\ R &= \sqrt{X^2 + Y^2} \end{aligned}$$

Since the particle is in equilibrium, the resultant force $R = 0$.

$$R = 0 \Rightarrow X = 0, \quad Y = 0 \quad (\text{since } X^2 \geq 0, Y^2 \geq 0)$$

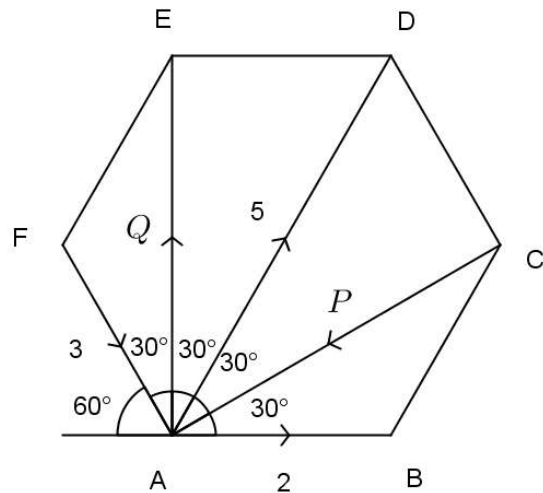
* It is necessary that the resolved components of the forces acting on the particle, in two different directions must be zero.

Example 7

ABCDEF is a regular hexagon. Forces of magnitudes $3, 5, P$ and Q act along CA, AD, AE and FA respectively. Find P and Q .

Resolving along AB

$$\begin{aligned} X &= 2 - P \cos 30^\circ + 5 \cos 60^\circ + 3 \cos 60^\circ \\ &= 2 - \frac{\sqrt{3}P}{2} + \frac{5}{2} + \frac{3}{2} \\ &= 6 - \frac{\sqrt{3}P}{2} \\ Y &= Q - P \sin 30^\circ + 5 \sin 60^\circ - 3 \sin 60^\circ \\ &= Q - \frac{P}{2} + \sqrt{3} \end{aligned}$$



Since the system of forces is in equilibrium

$$X = 0, \quad Y = 0$$

$$X = 0 \Rightarrow 6 - \frac{P\sqrt{3}}{2} = 0; \quad P = \frac{12}{\sqrt{3}} = 4\sqrt{3} \text{ N}$$

$$Y = 0 \Rightarrow Q - \frac{P}{2} + \sqrt{3} = 0$$

$$Q - 2\sqrt{3} + \sqrt{3} = 0$$

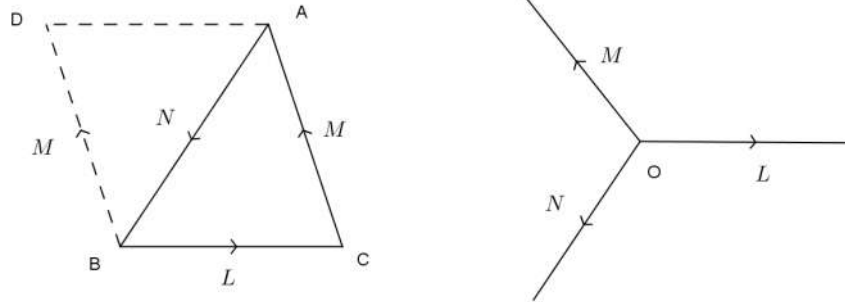
$$Q = \sqrt{3} \text{ N}$$

2.6 Three coplanar forces acting on a particle

1. Triangle of forces

If three forces, acting on a particle, can be represented in magnitude and direction by the sides of a triangle taken in order the forces will be in equilibrium.

Let L, M, N be three forces acting at O and represented by BC, CA, AB respectively (in magnitude and direction) taken in order of a triangle ABC , then L, M, N are in equilibrium.



Complete the parallelogram $BCAD$.

$$BD = CA, \quad BD \parallel CA$$

BD represents M in magnitude and direction.

Using **parallelogram law of forces**, the resultant R of L and M is represented by \overrightarrow{BA} .

i.e: $BA = N$ and direction of R is opposite to N .

Since $R = N$ and opposite in direction and act at O

Hence L, M, N are in equilibrium.

OR

$$\text{Using vectors } \overrightarrow{BC} + \overrightarrow{CA} = \overrightarrow{BA}$$

$$(\overrightarrow{BC} + \overrightarrow{CA}) + \overrightarrow{AB} = \overrightarrow{BA} + \overrightarrow{AB} = \underline{\underline{0}}$$

The vector sum of three forces is 0 . Since all three forces act at a point, they are in equilibrium.

2. Converse of triangle of forces

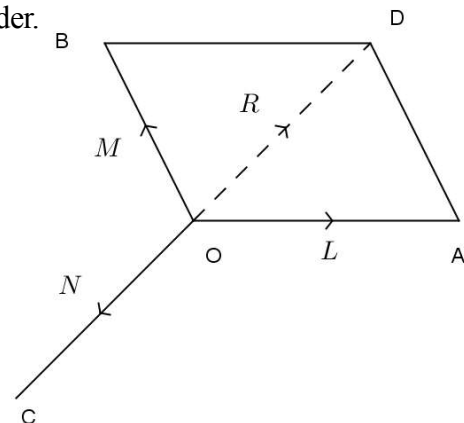
If three forces acting at a particle are in equilibrium. then they can be represented in magnitude and direction by the three sides of a triangle taken in order.

Let L, M and N be three forces acting at a particle and they are in equilibrium.

Let the three forces L, M, N acting at O

are represented by OA, OB, OC

(in magnitude and direction) respectively.



Complete the parallelogram OADB. Using parallelogram law of forces the resultant R of L and M is represented by OD. Since L, M and N are in equilibrium, R and N are in equilibrium.

Therefore $R = N$ and they are opposite in direction.

Hence in the triangle OAD, L is represented by OA, M is represented by AD and N is represented by DO

$$\begin{array}{l} \Delta OAD \\ \boxed{\begin{array}{l} L \rightarrow OA \\ M \rightarrow AD \\ N \rightarrow DO \end{array}} \end{array}$$

3. Lami's Theorem

If three forces acting at a particle are in equilibrium, each force is proportional to the sine of the angle between the other two.

If L, M, N are in equilibrium

$$\frac{L}{\sin \angle BOC} = \frac{M}{\sin \angle COA} = \frac{N}{\sin \angle AOB}$$

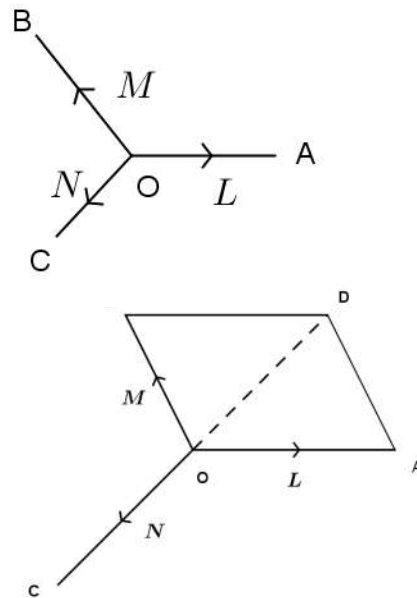
This theorem could be easily proved

Using sine rule for a triangle.

From the triangle of forces L, M, N can be represented by the sides of the triangle AOD. In the triangle AOD,

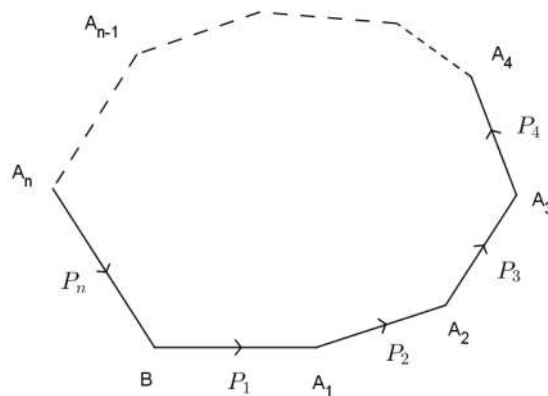
$$\frac{OA}{\sin \hat{O}DA} = \frac{AD}{\sin \hat{D}OA} = \frac{DO}{\sin \hat{O}AD}$$

$$\frac{L}{\sin \hat{B}OC} = \frac{M}{\sin \hat{C}OA} = \frac{N}{\sin \hat{A}OB}$$



4. Polygon of forces

If any number of forces acting on a particle, can be represented in magnitude and direction by the sides of a polygon taken in order, the forces will be in equilibrium.



$A_2A_3, \dots, A_{n-1}A_n, A_nB$ of a polygon $BA_1A_2A_3 \dots A_n$

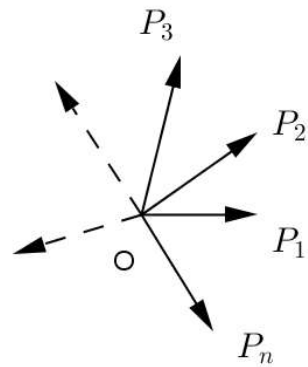
Then the forces are in equilibrium.

$$\vec{BA_1} + \vec{A_1A_2} = \vec{BA_2}$$

$$\vec{BA_1} + \vec{A_1A_2} + \vec{A_2A_3} = \vec{BA_2} + \vec{A_2A_3} = \vec{BA_3}$$

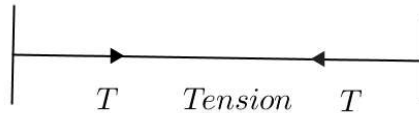
By vector addition,

$$\vec{BA_1} + \vec{A_1A_2} + \vec{A_2A_3} + \dots + \vec{A_{n-1}A_n} + \vec{A_nB} = 0$$



Tension of a string

A **light string** means that the weight of the string is negligible in comparison with the other weights in the given problem. The force which a string exerts on a body is called the tension and it acts along the string.



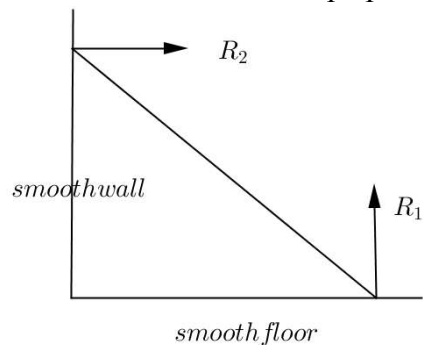
In a light string, the tension in the string is approximately the same throughout its length. If the string is heavy, the tension in the string varies from point to point.

Smooth surfaces

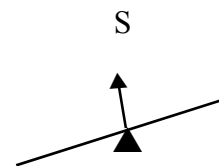
The only force acting between smooth bodies is normal reaction. This normal reaction is perpendicular to their common surface.

i.e., When there is a contact between smooth bodies, the normal reaction is perpendicular to the direction which the body is capable of moving.

The force between a rod and smooth floor is R_1 , perpendicular to floor. The force between a rod and smooth wall is R_2 , perpendicular to wall. R_1, R_2 are called normal reactions.



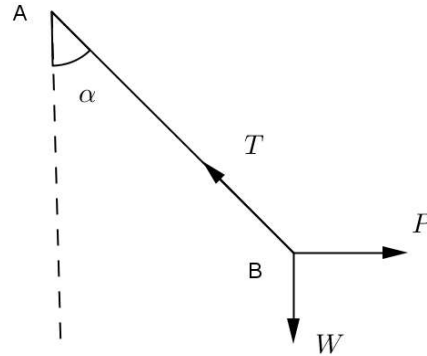
When a rod rests against a smooth peg the reaction S is normal to the rod.



2.7 Worked examples

Example 8

A particle of weight W is attached to one end B of a light string AB and hangs from a fixed point A . A horizontal force P is applied to the particle at B and rests in equilibrium with the string inclined at an angle α to the vertical. Find the tension in the string and the value of P in terms of W and α .



Method I

Forces acting on the particle.

- i. Weight W , vertically downwards
- ii. Force P , horizontally
- iii. Tension T , along the string

For equilibrium of the particle,

Resolving vertically,

$$\uparrow T \cos \alpha - W = 0 \Rightarrow T = \frac{W}{\cos \alpha}$$

Resolving horizontally,

$$\rightarrow P - T \sin \alpha = 0 \Rightarrow P = T \sin \alpha = W \tan \alpha$$

Method II (Triangle of forces)

Three forces T, W, P act on the particle and the particle is in equilibrium.

Consider the triangle BAC .

T can be represented by BA ; $T \longrightarrow BA$

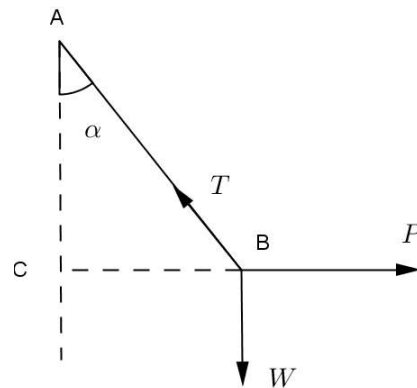
W can be represented by AC ; $W \longrightarrow AC$

P can be represented by CB ; $P \longrightarrow CB$

$$\frac{T}{BA} = \frac{W}{AC} = \frac{P}{CB}$$

$$\frac{T}{BA} = \frac{W}{AC}; \quad T = W \times \frac{BA}{AC} = \frac{W}{\cos \alpha}$$

$$\frac{W}{AC} = \frac{P}{CB}, \quad P = W \times \frac{CB}{AC} = W \tan \alpha$$

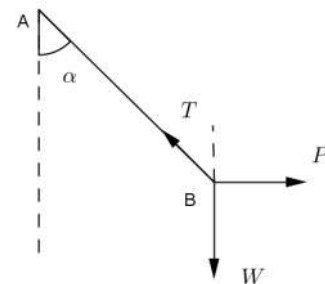


Method III (Lamis' Theorem)

$$\frac{T}{\sin 90} = \frac{W}{\sin(90+\alpha)} = \frac{P}{\sin(180-\alpha)}$$

$$\frac{T}{1} = \frac{W}{\cos \alpha} = \frac{P}{\sin \alpha}$$

$$T = \frac{W}{\cos \alpha}, \quad P = W \tan \alpha$$



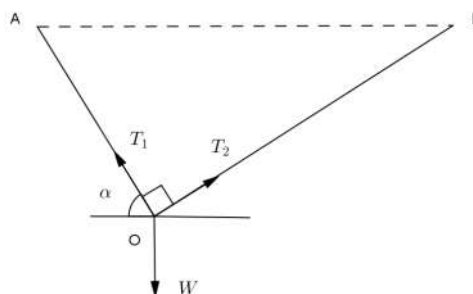
Example 9

A particle of weight W is attached to the ends O of two light strings OA and OB , each of length 50 cm, 120 cm respectively. The other ends A and B are attached to two points at the same level and the distance between A and B is 130 cm. Find the tensions in the string.

$$OA^2 + OB^2 = 50^2 + 120^2 = 130^2 = AB^2$$

Therefore $\angle AOB = 90^\circ$

$$\text{If } \angle AOB = \alpha, \cos \alpha = \frac{5}{13}, \sin \alpha = \frac{12}{13}$$



Forces acting at O

- i. Weight W , vertically downwards
- ii. Tension T_1 , along the string OA
- iii. Tension T_2 , along the string OB

Method I

For equilibrium of the particle.

Resolving horizontally

$$T_2 \cos(90 - \alpha) - T_1 \cos \alpha = 0$$

$$T_2 \sin \alpha - T_1 \cos \alpha = 0$$

$$12T_2 - 5T_1 = 0 \dots\dots\dots \textcircled{1}$$

Resolving vertically,

$$T_2 \sin(90 - \alpha) + T_1 \sin \alpha - W = 0$$

$$T_2 \cos \alpha + T_1 \sin \alpha - W = 0$$

$$5T_2 + 12T_1 = 13W \dots\dots\dots \textcircled{2}$$

From (1) and (2), $T_1 = \frac{12W}{13}$ and $T_2 = \frac{5W}{13}$

Method II (Triangle of forces)

AC is vertical. BO is produced to C .

consider the triangle OAC

$$T_1 \longrightarrow OA$$

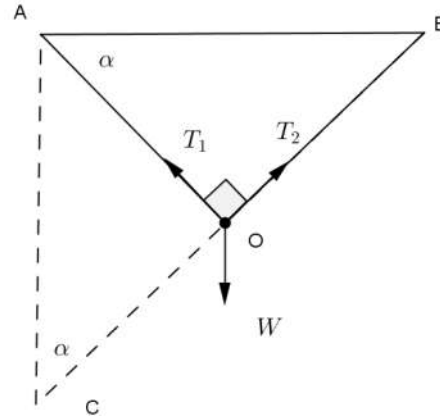
$$W \longrightarrow AC$$

$$T_2 \longrightarrow CO$$

$$\frac{T_1}{OA} = \frac{W}{AC} = \frac{T_2}{CO}$$

$$T_1 = W \cdot \frac{OA}{AC} = W \sin \alpha = \frac{12W}{13}$$

$$T_2 = W \cdot \frac{OC}{AC} = W \cos \alpha = \frac{5W}{13}$$



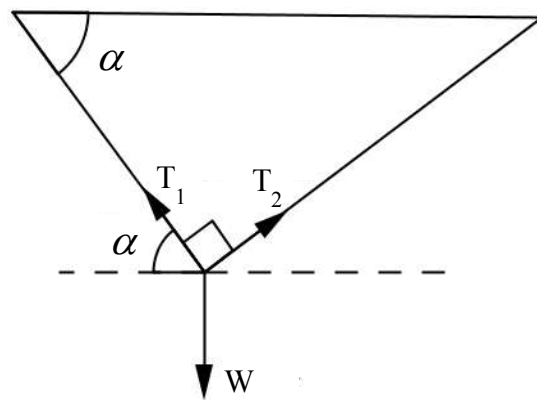
Method III (Lamis Theorem)

$$\frac{T_2}{\sin 90} = \frac{T_1}{\sin(180-\alpha)} = \frac{T_2}{\sin(90+\alpha)}$$

$$\frac{W}{1} = \frac{T_1}{\sin \alpha} = \frac{T_2}{\cos \alpha}$$

$$T_1 = W \sin \alpha = \frac{12W}{13},$$

$$T_2 = W \cos \alpha = \frac{5W}{13}$$



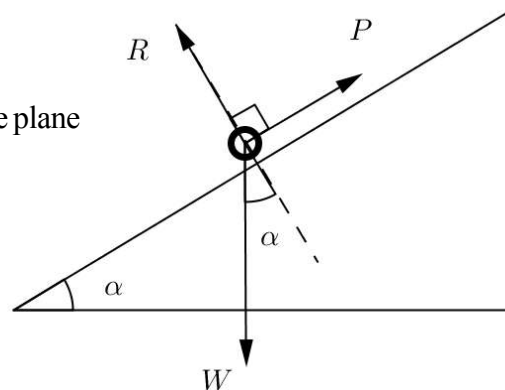
Example 10

A particle of weight W is placed on a smooth plane inclined at an angle α to the horizontal. Find the magnitude of the force

- i. acting along the plane upwards,
- ii. acting horizontally

to keep the particle in equilibrium

1. Forces acting on the particle are,
 - i. weight W , vertically downwards
 - ii. Normal reaction R , perpendicular to the plane
 - iii. Force P along the plane



Method I

For equilibrium of the particle,

Resolving along the plane,

$$\nearrow P - W \sin \alpha = 0; \quad P = W \sin \alpha$$

Resolving perpendicular to the plane,

$$\begin{aligned} \Sigma R - W \cos \alpha &= 0 \\ R &= W \cos \alpha \end{aligned}$$

Method II (Triangle of forces)

Consider the triangle ABC,

$$W \longrightarrow AB$$

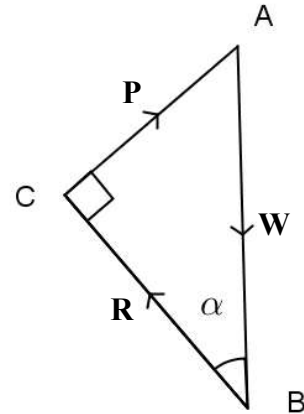
$$R \longrightarrow BC$$

$$P \longrightarrow CA$$

$$\frac{W}{AB} = \frac{R}{BC} = \frac{P}{CA}$$

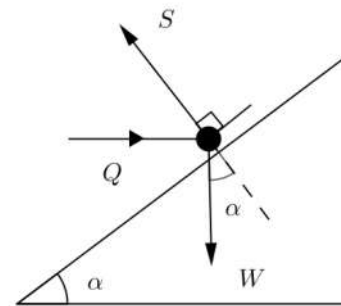
$$R = W \cdot \frac{BC}{AB} = W \cos \alpha$$

$$P = W \cdot \frac{CA}{AB} = W \sin \alpha$$



(ii). Forces acting on the particle

- i. Weight W , vertically downwards
- ii. Normal reaction S , perpendicular to the plane
- iii. Horizontal force Q



Method I

For equilibrium of the particle

resolving along the plane

$$\Sigma Q \cos \alpha - W \sin \alpha = 0$$

$$\Sigma M \quad Q = W \tan \alpha$$

$$S - W \cos \alpha - Q \sin \alpha = 0$$

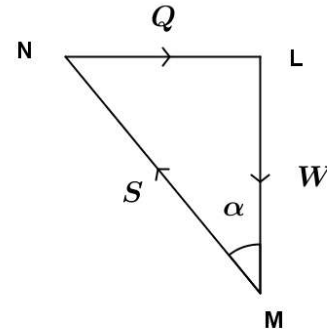
$$S = W \cos \alpha + Q \sin \alpha$$

$$= W \cos \alpha + \frac{W \sin^2 \alpha}{\cos \alpha} = W \left(\frac{\cos^2 \alpha + \sin^2 \alpha}{\cos \alpha} \right) = W \sec \alpha$$

Method II (Triangle of forces)

Consider the triangle LMN

- i) Weight $W \longrightarrow LM$
- ii) Normal reaction $S \longrightarrow MN$
- iii) Horizontal force $Q \longrightarrow NL$



$$\frac{W}{LM} = \frac{S}{MN} = \frac{Q}{NL}$$

$$Q = W \frac{NL}{LM} = W \tan \alpha$$

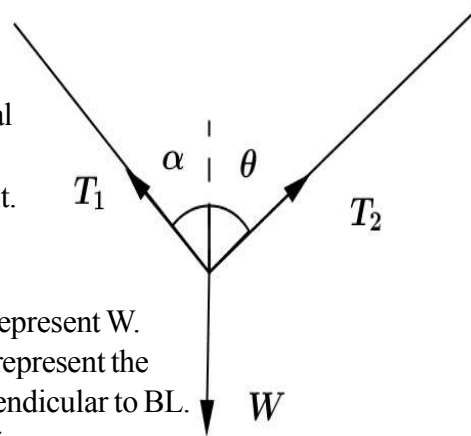
$$S = W \frac{MN}{LM} = \frac{W}{\cos \alpha} = W \sec \alpha$$

Example II

A particle of weight W is supported by two strings attached to it. If the direction of one string be at α ($0 < \alpha < \frac{\pi}{2}$) to the vertical, find the direction of the other string in order that its tension be minimum. In this case, find the tensions in both strings.

Forces acting on the particle are

- i) Weight of the particle W , vertically downwards,
 - ii) Tension T_1 in the string at an angle α with the vertical
 - iii) Tension T_2 in the other string T_2 should be minimum
- For the equilibrium of the particle, three forces act on it.



This could be done easily using triangle of forces.

Firstly we have to draw AB vertically downwards to represent W .

Then draw a line BL at an angle α with the vertical to represent the direction of T_1 .

For T_2 to be minimum, draw AC perpendicular to BL.

Now T_2 is represented by CA in magnitude and direction.

$$W \longrightarrow AB$$

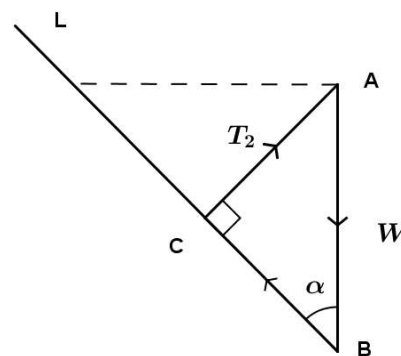
$$T_1 \longrightarrow BC$$

$$T_2 \longrightarrow CA$$

$$\frac{W}{AB} = \frac{T_1}{BC} = \frac{T_2}{CA}$$

$$T_1 = W \cos \alpha$$

$$T_2 = W \sin \alpha$$



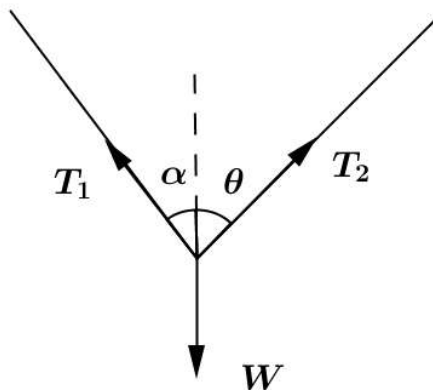
The direction of the second string (T_2) is perpendicular to the first string.

Method II

For equilibrium of the particle,
applying Lamis theorem

$$\frac{W}{\sin(\alpha + \theta)} = \frac{T_1}{\sin(180 - \theta)} = \frac{T_2}{\sin(180 - \alpha)}$$

$$T_1 = \frac{W \sin \theta}{\sin(\alpha + \theta)} \quad T_2 = \frac{W \sin \alpha}{\sin(\alpha + \theta)}$$



For T_2 to be minimum $\sin(\alpha + \theta)$ should be equal to 1. [i.e, $\sin(\alpha + \theta)$ should be maximum]

There fore $(\alpha + \theta) = \frac{\pi}{2}$

$$T_1 = W \sin \theta = W \sin \left[\frac{\pi}{2} - \alpha \right] = W \cos \alpha$$

$$T_2 = W \sin \alpha$$

The direction of T_2 is perpendicular to T_1

Example 12

The ends A and D of a light inextensible string ABCD are tied to two fixed points in the same horizontal line. Weights W and 3W are attached to the strings at B and C respectively. AB and CD are inclined to the vertical at angles 60° and 30° respectively. Show that BC is horizontal and find the tensions in the portions AB, BC and CD of the string.

Let BC be at an angle α with the horizontal.

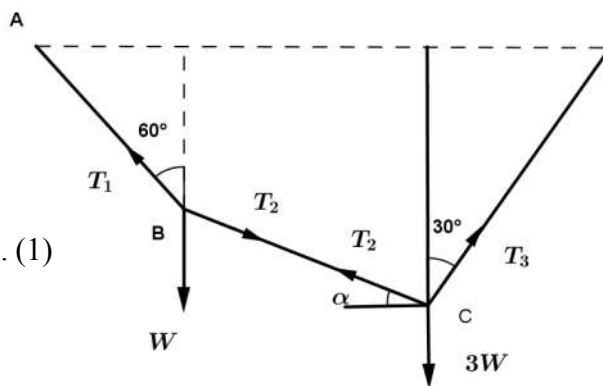
For equilibrium of B,
applying Lamis theorem

$$\frac{T_2}{\sin 120} = \frac{T_1}{\sin(90 - \alpha)} = \frac{W}{\sin(150 + \alpha)}$$

$$\frac{T_2}{\sin 60} = \frac{T_1}{\cos \alpha} = \frac{W}{\sin(30 - \alpha)} \quad \dots\dots (1)$$

For equilibrium of C,

$$\frac{T_2}{\sin 150} = \frac{T_3}{\sin(90 + \alpha)} = \frac{3W}{\sin(120 - \alpha)}$$



$$\frac{T_2}{\sin 30} = \frac{T_3}{\cos \alpha} = \frac{3W}{\sin(60+\alpha)} \dots\dots\dots(2)$$

From (1) and (2),

$$T_2 = \frac{W \sin 60}{\sin(30-\alpha)} = \frac{3W \sin 30}{\sin(60+\alpha)}$$

$$\sin 60 \cdot \sin(60+\alpha) = 3 \sin 30 \cdot \sin(30-\alpha)$$

$$\frac{\sqrt{3}}{2} \left[\frac{\sqrt{3}}{2} \cos \alpha + \frac{1}{2} \sin \alpha \right] = \frac{3}{2} \left[\frac{1}{2} \cos \alpha - \frac{\sqrt{3}}{2} \sin \alpha \right]$$

$$\sqrt{3} \cos \alpha + \sin \alpha = \sqrt{3} \cos \alpha - 3 \sin \alpha$$

$$4 \sin \alpha = 0; \quad \sin \alpha = 0; \quad \alpha = 0$$

Hence BC is horizontal

$$\text{From (1)} \quad T_1 = \frac{W}{\sin 30} = 2W$$

$$\text{Form (1)} \quad T_2 = \frac{W \sin 60}{\sin 30} = \sqrt{3} W$$

$$\text{Form (2)} \quad T_3 = \frac{3W}{\sin 60} = 2\sqrt{3}$$

Example 13

(a) Fores $F_1 = 4\mathbf{i} + 2\mathbf{j}$, $F_2 = 2\mathbf{i} - 5\mathbf{j}$ and $F_3 = -\mathbf{i} + \mathbf{j}$ act at a point. Find the magnitude and direction of the resultant of three forces.

(b) The coordinates of three points A, B and C are A(2,3), B(5,7) and C(-3,15)

i. Find the vectors \overrightarrow{AB} and \overrightarrow{AC} in terms of \mathbf{i} , \mathbf{j}

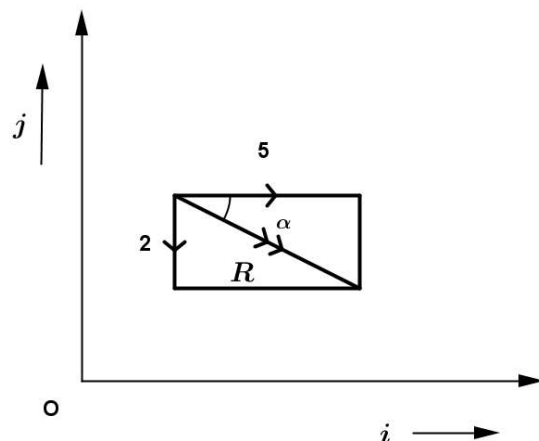
ii. Fores \mathbf{F}_1 and \mathbf{F}_2 of magnitudes 20 N and 65 N respectively act at the point A, along AB and AC respectively. Find the magnitude and direction of the resultant.

(unit vectors along the coordinate axes Ox and Oy are \mathbf{i} and \mathbf{j} respectively.)

$$\begin{aligned} \text{(a) } \mathbf{R} &= \mathbf{F}_1 + \mathbf{F}_2 + \mathbf{F}_3 \\ &= (4\mathbf{i} + 2\mathbf{j}) + (2\mathbf{i} - 5\mathbf{j}) + (-\mathbf{i} + \mathbf{j}) \\ &= 5\mathbf{i} - 2\mathbf{j} \end{aligned}$$

$$|\mathbf{R}| = \sqrt{5^2 + 2^2} = \sqrt{29}$$

$$\tan \alpha = \frac{2}{5}, \quad \alpha = \tan^{-1} \left(\frac{2}{5} \right)$$



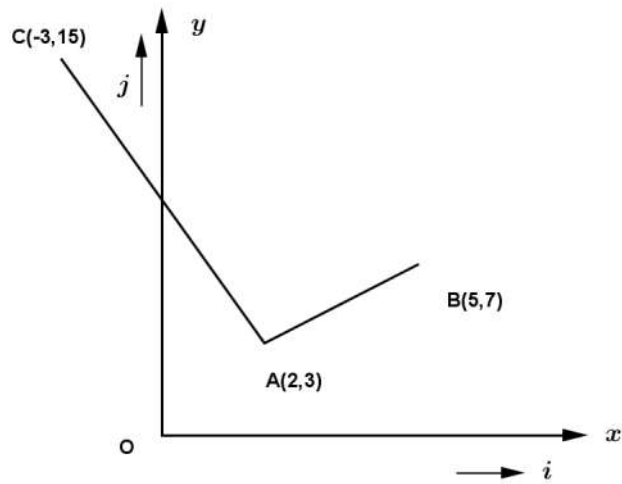
b) $A \equiv (2, 3), B \equiv (5, 7), C \equiv (-3, 15)$

$$\overline{OA} = 2\underline{i} + 3\underline{j}, \quad \overline{OB} = 5\underline{i} + 7\underline{j},$$

$$\overline{OC} = -3\underline{i} + 15\underline{j}$$

$$\begin{aligned} \overline{AB} &= \overline{OB} - \overline{OA} \\ &= (5\underline{i} + 7\underline{j}) - (2\underline{i} + 3\underline{j}) \\ &= 3\underline{i} + 4\underline{j} \end{aligned}$$

$$\begin{aligned} \overline{AC} &= \overline{OC} - \overline{OA} \\ &= (-3\underline{i} + 15\underline{j}) - (2\underline{i} + 3\underline{j}) \\ &= -5\underline{i} + 12\underline{j} \end{aligned}$$



Unit vector along \overline{AB} is $\frac{1}{5}(3\underline{i} + 4\underline{j})$

Unit vector along \overline{AC} is $\frac{1}{13}(-5\underline{i} + 12\underline{j})$

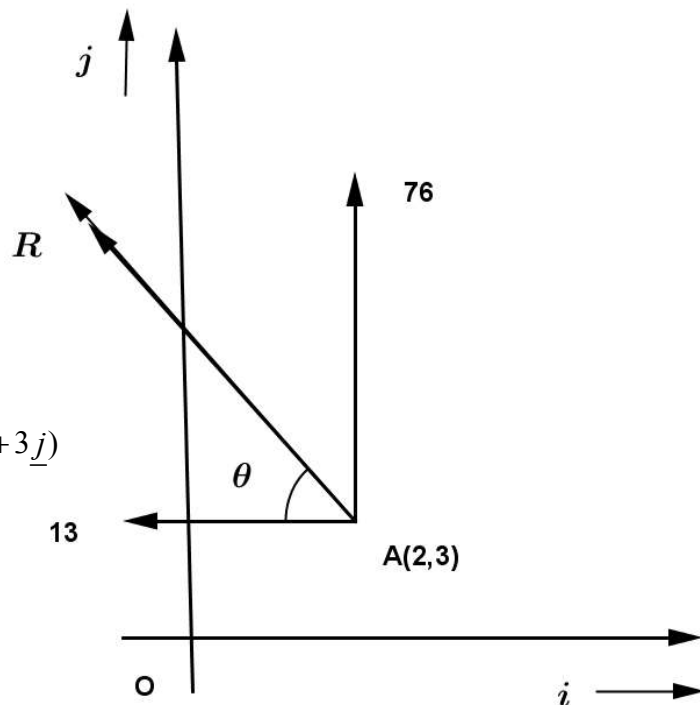
$$\begin{aligned} F_1 &= 20 \times \frac{1}{5} (3\underline{i} + 4\underline{j}) \\ &= 12\underline{i} + 16\underline{j} \end{aligned}$$

$$\begin{aligned} F_2 &= 65 \times \frac{1}{13} (-5\underline{i} + 12\underline{j}) \\ &= -25\underline{i} + 60\underline{j} \end{aligned}$$

$$\begin{aligned} \text{Resultant } \underline{R} &= \underline{F}_1 + \underline{F}_2 \\ &= (12\underline{i} + 16\underline{j}) + (-25\underline{i} + 60\underline{j}) \\ &= -13\underline{i} + 76\underline{j} \end{aligned}$$

$$|\underline{R}| = \sqrt{13^2 + 76^2}$$

$$\theta = \tan^{-1}\left(\frac{76}{13}\right)$$



2.8 Exercises

- Two forces P and Q act on a point at an angle θ . The resultant is R and α is the angle between R and P .
 - $P = 6, Q = 8, \theta = 90^\circ$; find R and α
 - $P = 10, Q = 8, \theta = 60^\circ$; find R and α
 - $P = 15, Q = 15\sqrt{2}, \theta = 135^\circ$; find R and α
 - $P = 8, R = 7, \theta = 120^\circ$; find Q and α
 - $P = 7, R = 15, \theta = 60^\circ$; find Q and α
- The forces F and $2F$ act on a particle. The resultant is perpendicular to F . Find the angle between the forces.
- The forces P and $2P$ Newton act on a particle. If the first be doubled and second be increased by 10 newtons, the direction of the new resultant is unchanged. Find the value of P .
- Two forces P and Q act on a particle at an angle θ . When θ is 60° , the resultant is $\sqrt{57}$ N and when θ is 90° the resultant is $5\sqrt{2}$ N. Find P and Q .
- If the resultant of two equal forces inclined at angle 2θ is twice the magnitude of the resultant when they at an angle 2α , show that $\cos\theta = 2\cos\alpha$.
- Two forces P and Q act at an angle θ . The resultant is equal to P in magnitude. When P is doubled the new resultant also equals to P in magnitude. Find Q in terms of P and the value of θ .
- The forces $P, P, \sqrt{3}P$ act on a particle and keep it in equilibrium. Find the angle between them.
- The resultant of two forces P and Q is $\sqrt{3}Q$ and makes an angle 30° with P . Show that $P = Q$ or $P = 2Q$.
- $ABCD$ is a square. Forces $P, 2\sqrt{2}P, 2P$ act at A along AB, AC, AD respectively. Find the resultant.

10. ABCD is a rectangle. $AB = 3\text{ m}$, $BC = 5\text{ m}$. Forces 6, 10, 12 newton act at A along AB, AC, AD respectively. Find the resultant.
11. ABCDEF is a regular hexagon. Forces $2\sqrt{3}$, 4, $8\sqrt{3}$, 2 and $\sqrt{3}$ newton act at B along BC, BD, EB, BF and AB respectively in the directions indicated by the order of the letters. Find the resultant.
12. ABCD is a square. E and F are the midpoints of BC and CD respectively. Forces 5, $2\sqrt{5}$, $5\sqrt{2}$, $4\sqrt{5}$, 1 newton act at A along AB, AE, CA, AF, AB respectively in the directions indicated by the order of the letters. Find the resultant.
13. ABCD is a square of 4 cm. The point E, F, G, H and J lie on the sides AB, BC, CD, DA respectively such that $AE = BF = CG = HD = DJ = 1\text{ cm}$. (Note that G and H lie on CD and $CG = 1\text{ cm}$, $GH = 2\text{ cm}$). Forces of magnitudes 10, $3\sqrt{10}$, $2\sqrt{5}$, 10, $\sqrt{10}$, 5 newton act at the point E in the directions EB, EF, EG, EH, EJ, EA respectively. Find the resultant of these forces.
14. ABC is an equilateral triangle and G is its centroid. Forces 10, 10 and 20 newton act at G along GA, GB and GC respectively. Find the magnitude and direction of their resultant.
15. A particle of weight 50 N is suspended by two light string of lengths 60 cm and 80 cm from two point at the same level and 100 cm apart. Find the tensions in the strings.
16. A particle of weight 100 N is placed on a smooth plane inclined at 60° to the horizontal. What force applied
 - (a) parallel to the surface of the plane
 - (b) horizontally
 will keep the particle at rest?
17. A particle of weight 30 N is suspended from two points A, B 60 cm apart and in the same horizontal line, by the strings of length 35 cm and 50 cm. Find the tension in each string.
18. A string of length 120 cm is attached to two points A and B at the same level at a distance of 60 cm apart. A ring of 50 N can slide freely along the string, is acted on by a horizontal force. F which holds it in equilibrium vertically below B. Find the tension in the string and the magnitude of F.
19. A string is tied to two points at the same level, and a smooth ring of weight W N can slide feely along the string is pulled by a horizontal force F N. In the equilibrium position, the portions of the strings at an angle 60° and 30° to the vertical. Find the value of F and tensions in the strings.

20. Ox, Oy are perpendicular axes and the unit vectors in the directions of Ox and Oy are \underline{i} and \underline{j} respectively.
- Forces $\underline{F}_1 = 3\underline{i} + 5\underline{j}$, $\underline{F}_2 = -2\underline{i} + \underline{j}$, $\underline{F}_3 = 3\underline{i} - \underline{j}$ act on a particle. Find the magnitude and direction of the resultant of \underline{F}_1 , \underline{F}_2 and \underline{F}_3 .
 - Forces $\underline{R}_1 = (2P\underline{i} - P\underline{j})$, $\underline{R}_2 = (-4\underline{i} + 3P\underline{j})$ and $\underline{R}_4 = (2Q\underline{i} - 5\underline{j})$ act on a particle and it is in equilibrium. Find the values of P and Q .
 - The coordinate of two points A and B are $(3, 4)$ and $(-1, 1)$ respectively. $2, 3, 5, 6\sqrt{2}$ newtons act at O , along Ox, Oy, OA, OB respectively. Express each force in the form $X\underline{i} + Y\underline{j}$ and hence calculate the magnitude and direction of the resultant of the four forces.
21. The unit vectors along rectangular cartesian axes Ox, Oy are $\underline{i}, \underline{j}$ respectively. Two forces P and Q acting on a particle are parallel to the vectors $4\underline{i} + 3\underline{j}$ and $-3\underline{i} - 4\underline{j}$ respectively. The resultant of the two forces is a force of magnitude 7N acting in the direction of vector \underline{i} . Calculate the magnitudes of P and Q .

3.0 Parallel Forces, Moments, Couples

3.1 Parallel Forces

In chapter two we have shown how to find the resultant of forces which act at a point. Now in this chapter we shall consider the action of parallel forces and the way to find their resultant.

Two types of parallel forces:

i. Like parallel forces

Two parallel forces are said to be like parallel forces when they act in the same direction (sense)

ii. Unlike parallel forces

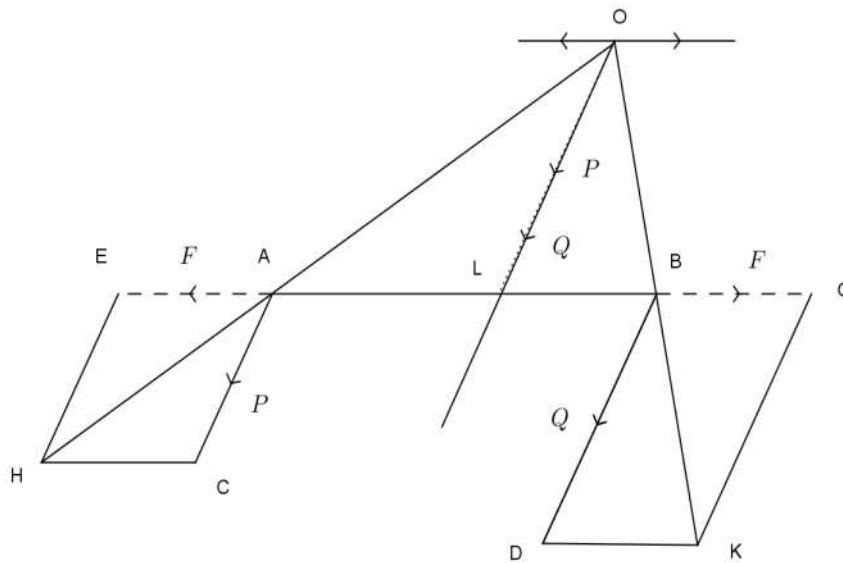
When two parallel forces act in the opposite parallel direction, they are said to be unlike.

Since parallel forces do not meet at a point their resultant cannot be obtained by direct application of parallelogram forces.

Resultant of two like parallel forces

Consider two like parallel forces P and Q acting at points A and B represented by lines AC and BD respectively.

At A and B introduce two equal and opposite forces F acting along the line AB as shown represented by AE and BG . These equal and opposite forces balance each other and have no effect on P and Q .



Complete the parallelograms AEHC and BDKG and produce the diagonals HA, KB to meet at O.

Draw OL parallel to AC (or BD) to meet AB at L.

The resultant of P and F at A, represented by AH and the resultant of Q and F at B, represented by BK may be supposed to act at O along OAH and OBK respectively.

These resultant forces may be resolved at O. The components are P along OL, F parallel to AE and Q along OL and F parallel to BG. Equal and opposite forces F at O balance each other. Hence the resultant of original forces P and Q is a forces (P + Q) parallel to original direction along OL.

Finding the position of L. The triangles OLA, ACH are similar.

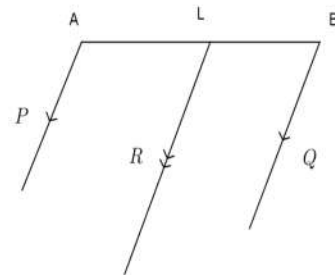
$$\frac{OL}{LA} = \frac{AC}{CH} = \frac{P}{F} \dots\dots\dots \textcircled{1}$$

and also the triangles OLB, BDK are similar.

$$\frac{OL}{LB} = \frac{BD}{DR} = \frac{Q}{F} \dots\dots\dots \textcircled{2}$$

From $\textcircled{1}$ and $\textcircled{2}$, $OL \times F = P \times LA = Q \times LB$

$$\frac{LA}{LB} = \frac{Q}{P}$$



ie. The point L divides AB internally in the ratio of the forces.

$$P \cdot AL = Q \cdot BL \text{ and the resultant } R = P + Q$$

Note : When $P = Q$, resultant R bisects AB.

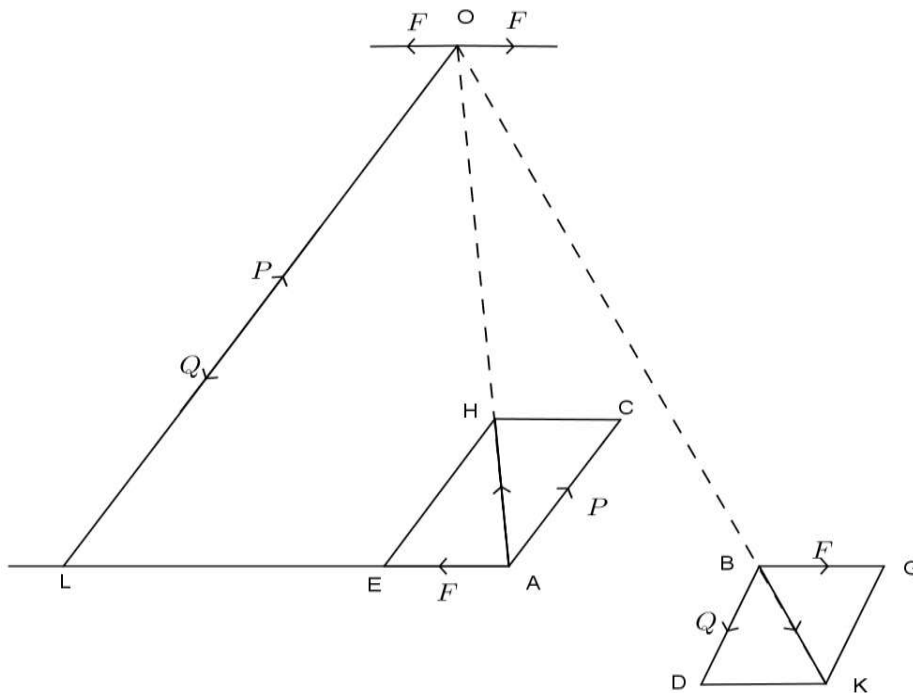
Case (ii)

Resultant of two unlike forces

Consider two unlike parallel forces P and Q ($P > Q$) acting at points A and B represented by AC and BD respectively.

At A and B introduce two equal and opposite forces F, acting along the line AB represented by AE and BG. They balance each other and have no effect on P and Q. Complete the parallelograms AEHC, BGKD and produce the diagonals AH, KB to meet at O. (They always meet at a point unless they are equal in magnitude, $P = Q$).

Draw OL parallel to CA (or BD) to meet BA produced at L.



The resultant of forces P and F at A, represented by AH and the resultant of forces Q and F at B, represented by BK may be supposed to act at O along AO and OB respectively. These resultant forces may be resolved at O. The components are P along LO, F parallel to AE and Q along OL, F parallel to BG. Equal and opposite forces at F balance each other. Hence the resultant of original forces P and Q is a single force (P - Q) acting along LO parallel to P in the direction of P.

The position of point L

by construction triangles OLA and HEA are similar

$$\frac{OL}{LA} = \frac{HE}{EA} = \frac{P}{F} \dots\dots\dots ①$$

and also the triangles OLB, BDK are similar.

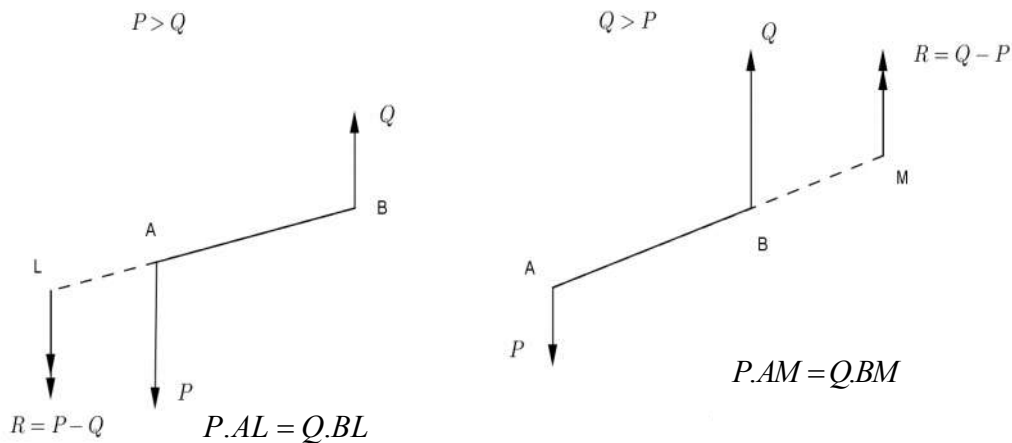
$$\frac{OL}{LB} = \frac{Q}{F} = \dots\dots\dots ②$$

From ① and ② $\frac{LA}{LB} = \frac{Q}{P}$

ie The point L divides AB externally in the inverse ratio of the forces

Note:

When P = Q, the triangles AEH and BDK are congruent so that diagonals AH and BK being parallel and will not meet at point O. Hence the construction fails, lead to the conclusion no single force is equivalent to two equal unlike parallel forces. Such a pair of forces constitutes a couple



will be discussed latter.

To find the resultant of any number of parallel forces.

- (i) If the forces are like parallel.

The resultant force can be obtained by the repeated application of finding the resultant of two like forces till all the forces have been taken.

The resultant will be the sum of all the forces and its direction is same as the direction of given forces if the forces are unlike.

- (i) If the forces are unlike paralls.

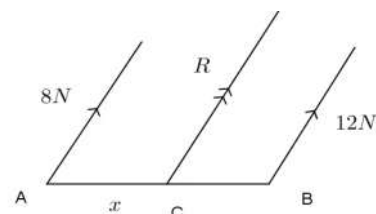
Divide the forces into two sets of like forces and find their resultant forces as mentioned above. Then find the resultant of a pair of unlike parallel forces as given below.

- a) If they are unequal, the resultant force is a single force with algebraic sum of the given forces as its magnitute.
- b) (i) If they are equal and the line of action are coincident, no resultant force and all the given forces are in equilibrium.
- (ii) If they are equal and line of action are not coincident they form a couple.

3.2 Worked examples

Example 1

- 1) Like parallel forces of 8 and 12 N act at points A and B where $AB = 15$ cm
- a. Find the magnitude of resultant and the point where the resultant cuts AB.
- b. When these forces are unlike find the resultant and the position of the line of action.



(a) $R = P+Q = 8+12 = 20N$

$8.AC = 12.BC$

$8x = 12(15-x)$

$20x = 12 \times 15$

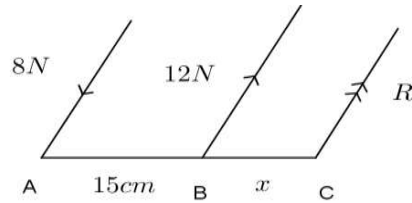
$AB = 9 \text{ cm}$

(b) $R = 12-8 = 4N$

$12x = (15+x)8$

$4x = 15 \times 8$

$x = 30 \text{ cm}$



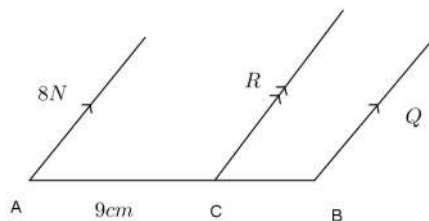
2) In the following examples A and B are the points where parallel forces P, Q acts and C be the point that the resultant R meets AB.

i. P and Q are like parallel forces, $P = 8 \text{ N}$, $R = 17 \text{ N}$, $AC = 9 \text{ cm}$

find Q and AB

ii. P, Q are unlike forces $P = 6 \text{ N}$, $AC = 18 \text{ cm}$, $CB = 16 \text{ cm}$

find Q and R



$P + Q = 17$

$Q = 17 - 8$

$= 9 \text{ N}$

$AC:CB = 9:8$

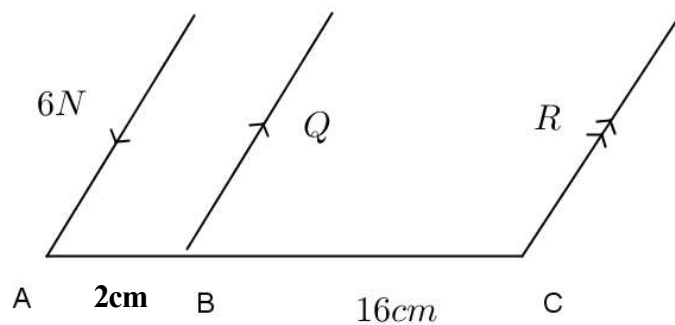
$6 \times 18 = Q \times 16$

$Q = \frac{27}{4}$

$R = Q - P$

$R = \frac{27}{4} - 6$

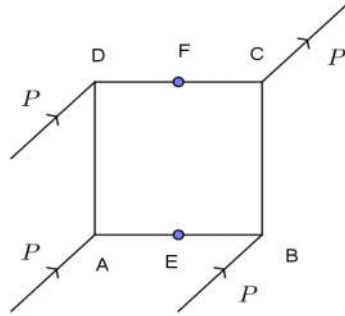
$R = \frac{3}{4}$



- 3) Four equal like parallel forces act at the vertices of a square show that the resultant passes through the centre of the square

Let the forces are PN.

Resultant of P at A and P at B is a like parallel force 2P at E when E is the midpoint of AB.



and resultant of P at C and P at D is also a like parallel force of magnitude 2P N acting at F. where F is the midpoint of CD.

Now resultant of two like parallel forces of 2P and 4P acting through the midpoint of EF which coincides with the centre of the square

Therefore resultant passes through the centre of the square.

- 4) P and Q are like parallel forces. If Q is moved parallel to itself through a distance x prove that the resultant of P and Q moves through a distance

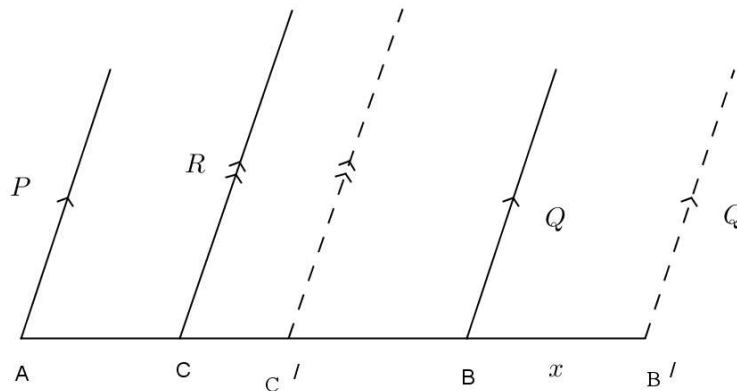
Let R be the resultant of forces P and Q acting at A and B and pass through the point C in AB

Then

$$\frac{AC}{CB} = \frac{Q}{P}$$

$$\frac{AC}{AB} = \frac{Q}{P+Q}$$

$$AC = \left(\frac{Q}{P+Q} \right) AB$$



Now Q moved a distance x then the resultant act at C' is AB

then

$$\frac{AC'}{C'B'} = \frac{Q}{P}$$

$$AC' = \left(\frac{Q}{P+Q} \right) AB' = \left(\frac{Q}{P+Q} \right) (AB + x)$$

Distance moved by the resultant

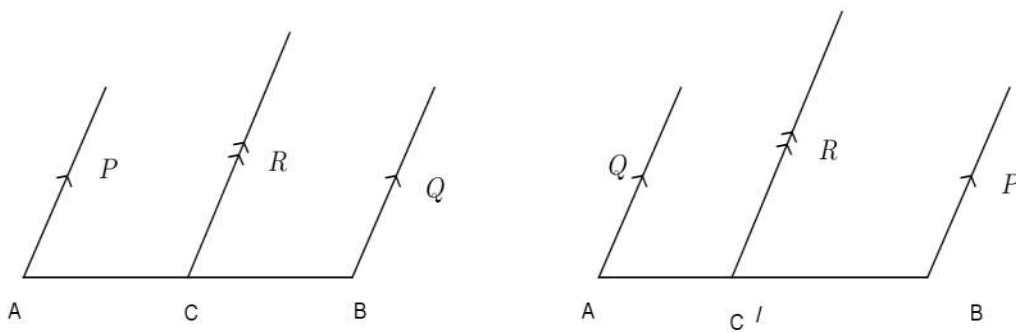
$$CC' = AC' - AC$$

$$CC' = \left(\frac{Q}{P+Q} \right) [AC + x - AC]$$

$$CC' = \left(\frac{Q}{P+Q} \right) x$$

- 5) Two like parallel forces P and Q act on a rigidbody at A and B respectively. If P and Q interchanged show that the point of the resultant cuts AB will move through a distance AB

will move through a distance $\left(\frac{P-Q}{P+Q} \right) AB$



$$\frac{AC}{CB} = \frac{Q}{P}$$

$$AC = \left(\frac{Q}{P+Q} \right) AB$$

$$\frac{AC'}{C'B} = \frac{P}{Q}$$

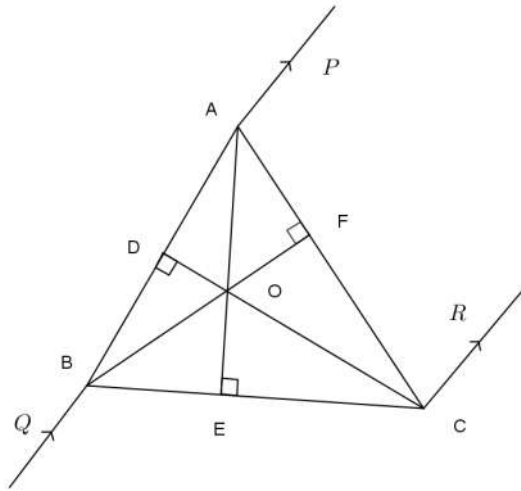
$$AC' = \left(\frac{P}{P+Q} \right) AB$$

$$AC' - AC = \left(\frac{P}{P+Q} \right) AB - \left(\frac{Q}{P+Q} \right) AB$$

$$= \left(\frac{P-Q}{P+Q} \right) AB$$

- 6) Like parallel forces P, Q, R act at the vertices of a triangle ABC. Show that if the resultant passes through the orthocentre of the triangle.

$$P : Q : R = \tan A : \tan B : \tan C$$



O be the orthocentre of the triangle

Given that the resultant passes through O.

Resultant of P and Q should pass through D, where $CD \perp AB$

$$\frac{AD}{DB} = \frac{Q}{P} = \frac{CD \cot A}{CD \cot B}$$

$$\frac{Q}{P} = \frac{\tan B}{\tan A} \quad \dots\dots\dots(1)$$

similarly resultant of Q and R should pass through E. Where $AE \perp BC$

$$\frac{BE}{EC} = \frac{R}{Q} = \frac{AE \cot B}{AE \cot C} = \frac{\tan C}{\tan B} \quad \dots\dots\dots(2)$$

$$(1), (2) \Rightarrow P : Q : R = \tan A : \tan B : \tan C$$

3.3 Exercises

1. Like parallel forces of magnitude 2, 5, 3 N act at the vertices A, B, C of a triangle ABC respectively. Where $AB = 4$ cm, $BC = 3$ cm and $AC = 5$ cm

Find i) Magnitude of the resultant
ii) The position of the line action of the resultant

2. Like parallel forces of magnitude P, P, 2P act at the vertices A, B, C of a triangle ABC. Show that the resultant passes through the midpoint of the line joining C to the midpoint of AB.

3. four equal like parallel forces act at the vertices of a square show that their resultant passes the centre of the square.

4. Three like parallel forces P, Q, R act at the vertices ABC of a triangle ABC. If the resultant passes through the incentre of the triangle prove that

$$\frac{P}{BC} = \frac{Q}{AC} = \frac{R}{AB}$$

5. Four forces are represented by \overline{AB} , $2\overline{BC}$, $3\overline{CD}$ and $4\overline{DA}$. Where ABCD is a square. Show that their resultant is represented in magnitude and direction by .

6. Two unlike parallel forces P and Q ($P > Q$) act at A and B respectively. If P and Q are increased by S. Show that the resultant will move by a distance $\frac{S \cdot AB}{P - Q}$

7. Three like parallel forces P, Q and R act at the vertices A, B, C of a triangle ABC. If the resultant passes through

(i) The centroid show that $P = Q = R$

(ii) The circumcentre show that $\frac{P}{\sin 2A} = \frac{Q}{\sin 2B} = \frac{R}{\sin 2C}$

8. Three parallel forces of magnitudes P, 2P, 3P act through the points A, B, C respectively on a straight line OABC where $OA = a$, $AB = b$ and $BC = c$. Show that the resultant act

through the point D in OABC where $OD = \frac{6a + 5b + 3c}{2}$

3.4 Moments

Forces acting on a rigidbody may tend to rotate the body, if one point of the body is fixed. The tendency of the force to turn the body introduces the idea of moment of a force about a point.

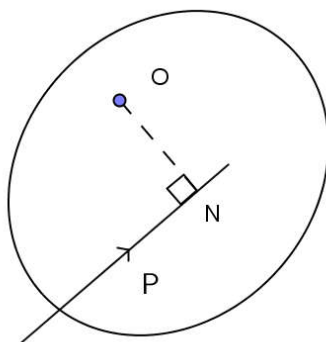
If a single force acts on a rigidbody which one point is fixed the force will tend to turn the body if the line of action of the force does not pass through that point.

Def :

The moment of a force about a given point is the product of the force and the perpendicular distance from the point to the line of action of the force.

Note :

When the line of action passes through the point O the moment about that point O is zero.



O is a fixed point on the body ON is the perpendicular drawn from O to the line of action of the force P. Then the moment of force P about O is $P \times ON$ and it tends to turn the body in the anticlockwise sense.

moment about O is = $P \times ON$ m

The SI unit of moment is Newton metre, Nm, moments are positive or negative according to the tendency of anticlockwise or clockwise rotation about the point.

When a certain number of forces acting on a body the algebraic sum of their moments is obtained by adding the moments of each force about that point with its sign.

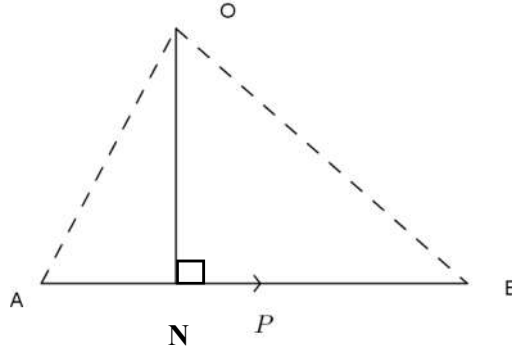
The moment of a force is a vector quantity as it has magnitude and direction (sense)

Graphical representation of moment

Suppose the force P is represented by line segment AB in magnitude and direction. Let O be a point about which moment to be taken. ON is the perpendicular from O to AB, then the moment of force P about O is $P \times ON$ m = $AB \times ON$

But area of triangle OAB = $\frac{1}{2} AB \times ON$

Hence twice the area of the triangle AOB, whose base represents the force and vertex is the point about which moment to be taken is numerically equal to the moment of the force about that point. Hence $P \cdot ON = 2\Delta OAB$



Note:

The graphical representation is used to prove some fundamental theorems about moment.

Varignon’s Theorem

The algebraic sum of moments of any two coplaner forces about any point in their plane is equal to the moment of their resultant about the same point. We have two cases to consider

- (i) forces are non parallel
- (ii) forces are parallel

case (i) When the forus are non parallel.

Proof : When forces meet at a point. Let P and Q be the forces acting at A and O be the point in their plane and the moment is to be taken about O. Draw OC parallel to the direction of P to meet the line of action of Q at C.

Let AC represents Q in magnitude and on the same scale AB to represent force P.

Complete the parallelogram ABCD. Join OA and OB. AD represents the resultant R of P and Q.

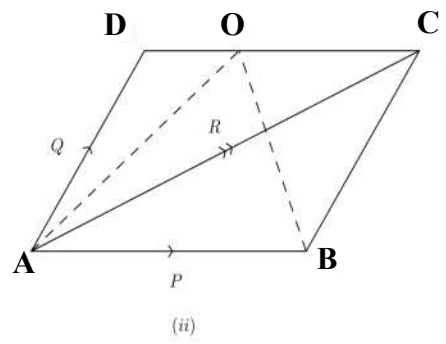
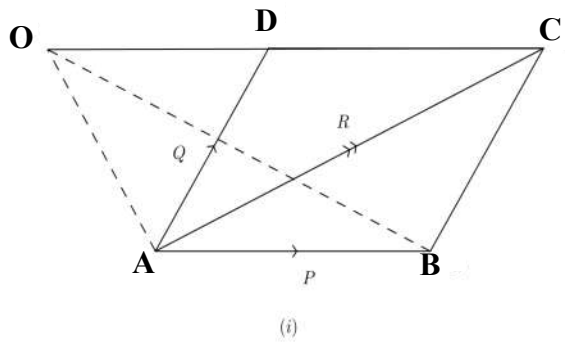
O have two passibilities as shown above

In both, moment of P about O is $m \cdot 2\Delta OAB$

moment of Q about O is $m \cdot 2\Delta OAD$

and moment of R about O is $2\Delta OAC$

$$\begin{aligned}
 \text{In figure (i) sum of the moments of P and Q is} &= 2\Delta OAB + 2\Delta OAD \ m \\
 &= 2\Delta ABC + 2\Delta OAD \\
 &= 2\Delta ACD + 2\Delta OAD \ m \\
 &= 2\Delta OAC \ m \\
 &= \text{moment of R about O} \ m
 \end{aligned}$$

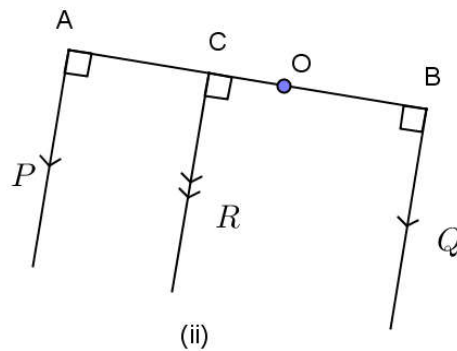
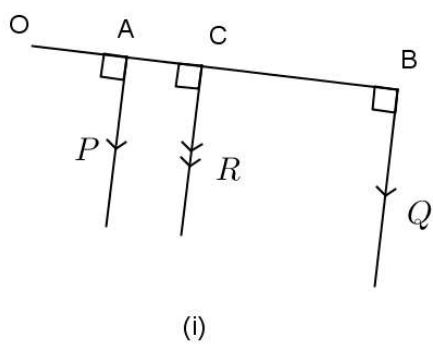


In figure (ii)

sum of the moments of P and Q is

$$\begin{aligned}
 &= \Delta OAB - 2\Delta AOD \text{ m} \\
 &= 2\Delta ADC - 2\Delta AOD \text{ m} \\
 &= 2\Delta AOC \text{ m} \\
 &= \text{moment of R about O}
 \end{aligned}$$

case (ii) When the forces are parallel



Let P and Q be two like parallel forces acting and O be any point in their plane as shown above
 Draw OAB perpendicular to the forces to meet their lines of action at A and B.

Let R be the resultant of P and Q and acts through C, where OC is perpendicular to R and $AC : CB = Q : P$

In figure (i) sum of the moments of P and Q about O = $P \times OA + Q \times OB$ ○

$$\begin{aligned}
 &= P(OC - AC) + Q(OC + CB) \text{ ○} \\
 &= (P + Q)OC - P \times AC + Q \times CB \text{ ○}
 \end{aligned}$$

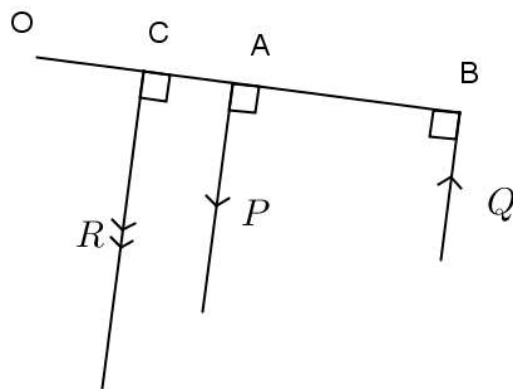
Since $\frac{AC}{CB} = \frac{Q}{P}$

$$P \times AC = Q \times CB$$

$$\begin{aligned} \text{sum of the moments} &= (P + Q) \times OC \circ \\ &= \text{moment of R about O.} \end{aligned}$$

In figure (ii) sum of the moments of P and Q is = $P \times OA_m + Q \times OB_o$

$$\begin{aligned} &= P \times OA - Q \times OB_m \\ &= P(OC + CA) - Q(CB - OC) \\ &= (P + Q)OC + P \times AC - Q \times CB \\ &= (P + Q) OC_m \\ &= \text{moment of R about O.} \end{aligned}$$



When forces are unlike and parallel

Let P, Q be unlike parallel forces and $P > Q$

$$\text{then } R = P - Q$$

sum of the moment about O

$$\begin{aligned} &= P \times OA - Q \times OB \\ &= P(OC + CA) - Q(OC + CB) \\ &= (P - Q)OC + P \times AC - Q \times CB \\ &= (P - Q) OC \circ \\ &= \text{moment of R about O.} \end{aligned}$$

Note : The algebraic sum of moments about any point in the line of action of their resultant is zero.

Generalised Theorem

This is known as principle of moments. If any number of coplaner forces acting on a rigidbody has a resultant, the algebraic sum of moments of the forces about any point in their plane is equal to the moment of their resultant about that point.

If a system of coplaner forces is in equilibrium their resultant is zero and its moment about any point in their plane must be zero.

If a system of coplaner forces is in equilibrium then the algebraic sum of their moment about any point in their plane is zero.

The converse, is not true.

If the sum of moments of a system of a coplaner forces about one point in their plane is zero does not mean the system of forces is in equilibrium, for the point may lie on the line of action of the result.

3.5 Worked examples

Example 7

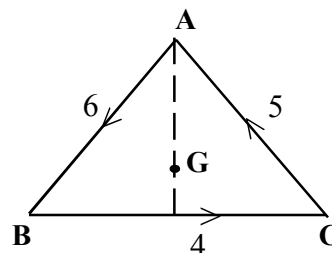
Forces of 4, 5, 6N act along the sides BC, CA and AB of an equilateral triangle ABC of side 2 m in the direction indicated by the order of letters. Find the sum of their moments about the centroid of the triangle.

Let G be the centroid $AD = 2\sin 60$

$$= 2 \frac{\sqrt{3}}{2} = \sqrt{3}m$$

$$GD = GE = GF = \frac{1}{3}\sqrt{3}m$$

$$\begin{aligned} \text{Sum of moments about G} &= 4 \times \frac{1}{\sqrt{3}} + 5 \times \frac{1}{\sqrt{3}} + 6 \times \frac{1}{\sqrt{3}} \\ &= \frac{15}{\sqrt{3}} m = 5\sqrt{3} \text{ Nm m} \end{aligned}$$



Example 8

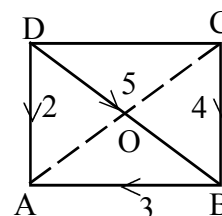
The side of a square ABCD is 24m. Forces of 4, 3, 2 and 5N act along CB, BA, DA and DB respectively as indicated by the order of letters. Find the sum of their moments about

- (i) vertex C (ii) The centre of the square O

$$CO = 4\cos 45 = \frac{4}{\sqrt{2}} = 2\sqrt{2}$$

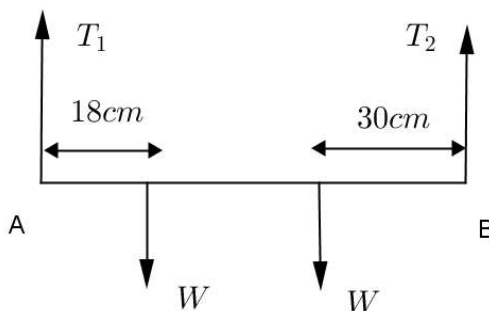
$$\begin{aligned} \text{Sum of moments about C} &= 2 \times 4 - 3 \times 4 + 5 \times 2\sqrt{2} \text{ m} \\ &= (10\sqrt{2} - 4) \text{ Nm} \end{aligned}$$

$$\begin{aligned} \text{Sum of moments about O} &= 4 \times 2 + 3 \times 2 - 2 \times 2 \text{ m} \\ &= 10 \text{ Nm} \end{aligned}$$



Example 9

A light rod of 72 cm has equal weights attached to it, one at 18 cm from one end and other at 30 cm from other end. The rod is suspended in a horizontal position by two vertical strings attached to the ends of the rod. If the strings can just support a tension of 50 N find the magnitude of the greatest weight that can be placed.



Let the equal weight be W and the tension in the strings be T_1, T_2 N.

for equilibrium of the rod $\uparrow T_1 + T_2 - 2W = 0$

Bm $-T_1 \times 72 + W \times 54 + W \times 30 = 0$

$$72T_1 = 84W$$

When T_1 is maximum ($T_1 = 50$)

$$72 \times 50 = 84W$$

$$W = \frac{72 \times 50}{84} = 42\frac{6}{7} \text{ N}$$

Am $T_2 \times 72 - W \times 18 - W \times 42 = 0$

$$72T_2 = 60W, \text{ When } T_2 \text{ is maximum } (T_2 = 50)$$

$$W = \frac{72 \times 50}{60} = 60 \text{ N}$$

Therefore the greatest weight can be placed is $42\frac{6}{7}$ N

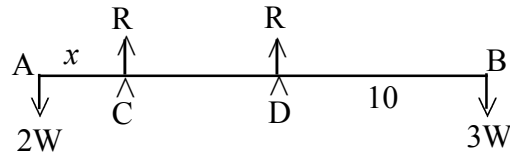
Example 10

A light rod of AB 20cm long rests on two pegs whose distance apart is 10 cm. Weights of $2W$ and $3W$ are suspended from A and B. Find the position of the pegs so that the reaction of the pegs be equal.

Let the distance of one from A is x cm.

The rod is in equilibrium. Therefore sum of moments taking moments about C, is zero.

$$\begin{aligned} R \times 10 + 2Wx - 3W(20 - x) &= 0 \\ 10R &= 60W - 5Wx \dots\dots\dots (1) \end{aligned}$$



taking moments about O

$$R \times 10 + 3W(10 - x) - 2W(10 + x) = 0$$

$$10R = 5Wx - 10W \dots\dots\dots(2)$$

(1) & (2) $10x = 70$

$$x = 7$$

Distance of pegs from A is 7cm & 17cm

Example 11

The side of a regular hexagon ABCDEF is $2m$. Forces of 1, 2, 3, 4, 5, 6N act along the sides AB, CB, DC, DE, EF and FA respectively in the order of letters. Find the sum of their moments about

- (i) Vertex A (ii) centr O the hexagon

$$AL = 2\sin 60$$

$$= \sqrt{3}m$$

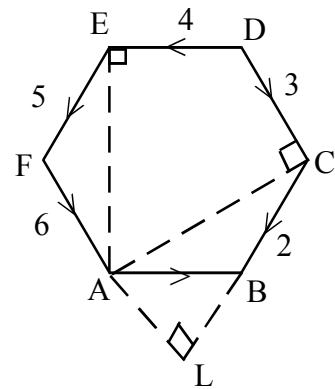
Sum of moments about A \circ

$$= 2 \times \sqrt{3} + 3 \times 2\sqrt{3} - 4 \times 2\sqrt{3} - 5 \times \sqrt{3}$$

$$= -5\sqrt{3} \circ$$

$$= 5\sqrt{3} \text{ Nm m}$$

$$OM = 2\sin 60 = \sqrt{3}m$$



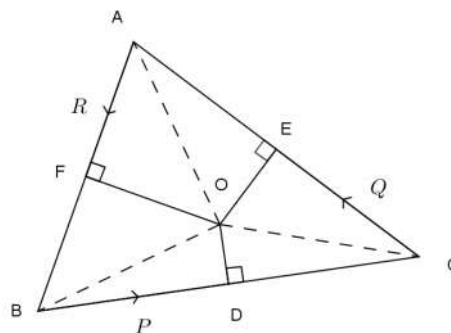
Sum of moments about O $m = 1 \times \sqrt{3} - 2 \times \sqrt{3} - 3 \times \sqrt{3} + 4 \times \sqrt{3} + 5 \times \sqrt{3} + 6 \times \sqrt{3}$

$$= 11\sqrt{3} \text{ Nm}$$

Example 12

Three forces P, Q, R act in the same sense along the sides BC, CA, AB of a triangle ABC. If the resultant passes through the circumcentre of the triangle show that

$$P \cos A + Q \cos B + R \cos C = 0$$



$$\hat{BOD} = \hat{A}, \hat{COE} = \hat{B}$$

Let R' be the radius. Then $R' = OA = OB = OC$. *of* circumcentre and $\hat{AOF} = \hat{C}$

Taking moment about O

$$P \times OD + Q \times OE + R \times OF = 0$$

$$P \cdot OB \cos A + Q \cdot OC \cdot \cos B + R \cdot OA \cos C = 0$$

since $OB = OC = OA$

$$P \cos A + Q \cos B + R \cos C = 0$$

3.6 Exercises

- Masses of 1, 2, 3, 4 kg are suspended from a uniform rod of length 1.5 m and mass 3 kg at distances of 0.3 m, 0.6 m, 0.9 m, 1.2 m from one end. Find the position of the point about which the rod will balance.
- A uniform beam AB of 3m long and mass 6 kg is supported at A and at another point on the rod. A load of 1 kg is suspended at B, load of 5 kg and 4 kg at points 1 m and 2 m from B. If the pressure on support A is 40 N, find the position of the other support.
- A uniform bar of 0.6 m long and of mass 17 kg is suspended by two vertical strings. One is attached at a point 7.5 cm from one end and just can support a weight of 7 kg without breaking it and other string is attached 10 cm from other end and can just support 10 kg without breaking it. A weight of mass 1.7 kg is now attached to the rod. Find the limits of the positions in which it can be attached without breaking either string.
- ABCD is a square of side a . Forces of 2, 3, 4 N act at A along AB, AD and AC respectively. Find the point where the line of action of the resultant meet DC.
- Three forces P, Q, R acting at the vertices A, B, C respectively of a triangle ABC each perpendicular to the opposite side and in equilibrium. Show that $P : Q : R = a : b : c$
- Three forces P, Q, R act along the sides BC, CA and AB of a triangle ABC. If their resultant passes through the centroid show that

$$(i) \quad \frac{P}{\sin A} + \frac{Q}{\sin B} + \frac{R}{\sin C} = 0 \quad (ii) \quad \frac{P}{BC} + \frac{Q}{CA} + \frac{R}{AB} = 0$$

- The resultant of three forces act in the same sense along the sides BC, CA, AB of a triangle ABC passes through the orthocentre and circumcentre.

Prove that
$$\frac{P}{(b^2 - c^2) \cos A} = \frac{Q}{(c^2 - a^2) \cos B} = \frac{R}{(a^2 - b^2) \cos C}$$

- A system consists of three forces P, λP , $\lambda^2 P$ acting along the lines BC, CA, AB in the sense indicated by the order of the letters. Show that if the resultant passes through the orthocentre of the acute angled triangle ABC then

$$\frac{1}{\cos A} + \frac{\lambda}{\cos B} = \frac{\lambda^2}{\cos(A+B)}$$

Deduce that λ is necessarily negative.

3.7 Couples

Definition : Two equal unlike parallel forces whose line of action are not the same form a couple.

The effect of a couple is causing rotation.

Couples are measured by their moments.

The perpendicular distance between the two lines of action is called arm of the couple.

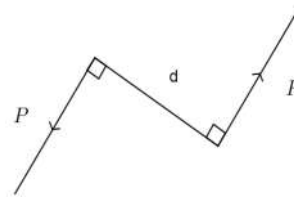
moment of a couple

The moment of a couple is the product of one of the forces and arm of the couple.

ie, moment of a couple = magnitude of a force \times distance between them.

$$M = P \times d$$

$$= Pd \text{ m}$$



A couple is said to be positive or negative according to its tendency to cause anticlockwise or clockwise rotation.

Theorem

The algebraic sum of the moments of the two forces forming a couple about any point in their plane is constant and equal to the moment of the couple.

Proof

Let forces of the couple equal to P and O be any point in their plane. Draw OAB perpendicular to the lines of action of forces to meet at A and B.

Algebraic sum of the moments about O

$$= P \times OB - P \times OA$$

$$= P \times AB \text{ m}$$

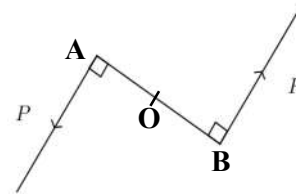
$$= \text{moment of the couple}$$

Sum of the moments about O

$$= P \times OA + P \times OB$$

$$= P(OA + AB)$$

$$= P(AB) \text{ m}$$



Therefore moment of a couple is same whatever the point O is taken

ie, moment of a couple is independent of the position of the point.

Theorem

Two couples acting in one plane of a rigidbody are equivalent to a single couple whose moments is the algebraic sum of the moments of the two couples.

Two cases to consider

Proof

Case (i) When the lines of action of forces are parallel let $(P, P), (Q, Q)$ be the forces of the couples acting as shown in the diagram and $OABCD$ is perpendicular from the point O to their lines of action. Resultant of forces P and Q acting at A and F is a force $(P + Q)$ acting at E where $AE : EF = Q : P$ and resultant of forces P and Q acting at B and D is $(P + Q)$ acting at C where $BC : CD = Q : P$. Now equal parallel and dislike forces $(P + Q)$ forms a single couple, which is the resultant couple of the two couple.

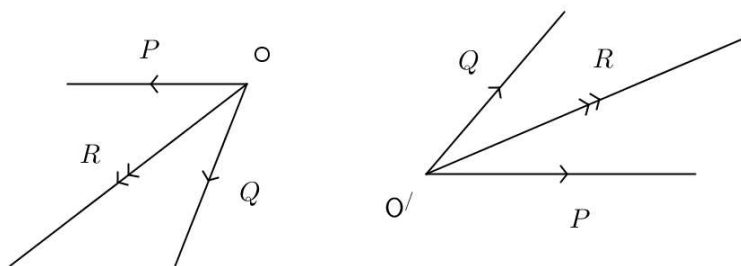
$$\begin{aligned}
 \text{moment of the couple} &= \text{sum of the moments of forces } (P + Q) \text{ at } E \\
 &\quad \text{and } (P + Q) \text{ at } C \text{ about } O \\
 &= \text{sum of the moment of } P \text{ at } A \text{ and } P \text{ at } B \text{ and} \\
 &\quad Q \text{ at } F \text{ and } Q \text{ at } D \\
 &= \text{sum of the moment of the given couple}
 \end{aligned}$$

case (ii) When the lines of action of forces are not parallel

Let P, P, Q, Q be the forces of the couples and one of the force P and one of the force Q meet at O as shown in the diagram and other forces P and Q meet at O' .

Forces P and Q at O has a resultant R at O and forces P and Q at O' has a resultant R at O'

Their resultant forces are equal parallel and dislike forces form a couple.



$$\begin{aligned}
 \text{Moment of the couple} &= \text{Moments of } R \text{ at } O' \text{ about } O \\
 &= \text{sum of the moments of } P \text{ at } O' \text{ and } Q \text{ at } O' \\
 &= \text{sum of the moments of given couples}
 \end{aligned}$$

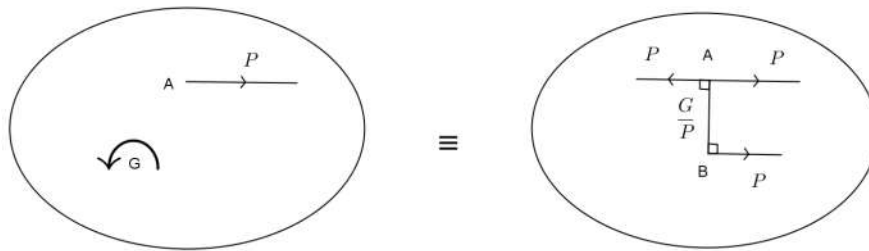
We can deduce the following

1. Two couples acting in a plane whose moments are equal and opposite balance each other.
2. Any two couples of equal moment and in the same plane are equivalent.

Theorem

Resultant of a force and a couple in the same plane, A force and a couple acting in the same plane on a rigid body are equivalent to a single force equal and parallel to the given force acting on another point.

Proof



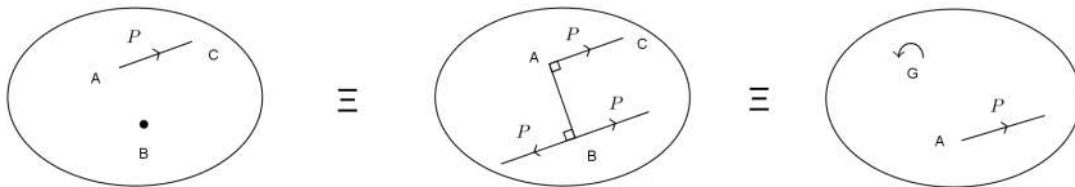
Let **P** be the single force acting at A and G be the couple acting in the same plane.

G can be replaced by two forces P and P acting at A and another point B where $AB = \frac{G}{P}$

Now equal and opposite forces P, P at A balance each other so that the resultant is single force P at B.

Theorem

A force acting at any point of A rigidbody is equivalent to an equal and like parallel force acting at any other point together with a couple.



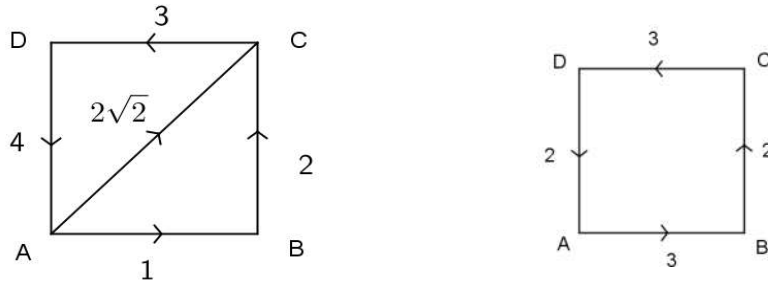
Proof

Let **P** be the given force acting at A along AC and B be any other point. Let perpendicular distance from B to AC is d. Introduce equal and opposite parallel forces P at B. One of these forces with opposite to the P at A forms a couple $G = P \times d$ and other force at B in the single force P.

3.8 Worked examples

Example 13

ABCD is a square of side 1 m. Forces of magnitude 1, 2, 3, 4, $2\sqrt{2}$ N act along the sides AB, BC, CD, DA and diagonal AC of the square ABCD in the given order. Show that the resultant is a couple and find its moment.

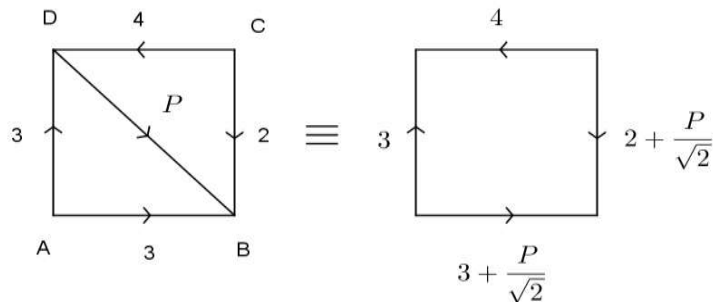


Resolve $2\sqrt{2}$ N along AD and AB. The components are $2\sqrt{2} \cos 45 = 2\text{N}$.

Now the system is equivalent to forces acting along the sides as shown above. The system consists of two sets of parallel, equal unlike forces forming two couples, and can be combined as a single couple of moment $3 \times 1 + 2 \times 1 = 5 \text{ Nm}$.

Example 14

ABCD is a square of forces of magnitude 3, 2, 4, 3, P N act along AB, CB, CD, AD and DB respectively indicated by the order of letters. If the system reduces to a couple find the value of P.



Resolve P N along AB and CB, as $P \cos 45 = \frac{P}{\sqrt{2}}$ N.

To reduce to a couple, $3 + \frac{P}{\sqrt{2}} = 4$ and $2 + \frac{P}{\sqrt{2}} = 3$

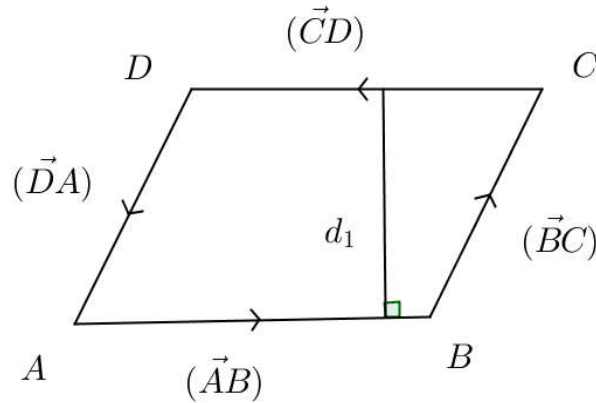
$$\frac{P}{\sqrt{2}} = 1 \text{ and } \frac{P}{\sqrt{2}} = 1$$

$$P = \sqrt{2} \text{ and } P = \sqrt{2}$$

$$\therefore P = \sqrt{2}$$

Example 15

ABCD is a parallelogram Forces represented by \vec{AB} , \vec{BC} , \vec{CD} , \vec{DA} act along the sides respectively in the order. Show that they are equivalent to a couple with moment numerically equal to twice the area of the parallelogram.



Forces \vec{AB} and \vec{CD} are equal opposite and parallel forces or couple of moment $AB \times d_1$.
When d_1 is the distance between AB and CD.

Forces \vec{BC} and \vec{DA} are also equal opposite and parallel forms another couple of moment $BC \times d_2$

also both couple acts in the same sense, so the moment of the resultant couple is

$$AB \times d_1 + BC \times d_2$$

But $AB \times d_1 = BC \times d_2 = \text{area of the parallelogram}$

Hence the moment of the couple equal to twice the area of the parallelogram.

3.9 Exercises

1. ABCD is a square of side $2m$. Forces a, b, c and d act along AB, BC, CD and DA taken in order and forces $p\sqrt{2}, q\sqrt{2}$ act along AC and BD respectively. Show that if $p+q = c - a$ and $p - q = d - b$ the forces are equivalent to a couple of moment $a + b + c + d$.

2. P and Q are two unlike parallel forces. If a couple with each forces F and whose arm is a in the plane of P and Q is combined with them. Show that the resultant is displaced through a distance $\frac{Fa}{(P+Q)}$

3. If three forces P, Q and R acting in the verticle of a triangle ABC along the tangents (in the same letters) to the circumcircle are equivalent to a couple. Show that

$$P : Q : R = \sin 2A : \sin 2B : \sin 2C$$

4. ABCD is a square D and E are the midpoints of CD and BC respectively. Forces P, Q, R act along AD, DE and EA in the direction indicated by the order of letters. If the system reduces to a couple show that $P : Q : R = \sqrt{5} : \sqrt{2} : \sqrt{5}$

5. Forces 4, 3, 3 N act along the sides AB, BC, CA of a triangle ABC of an equilateral triangle ABC repectively of side $0.6m$ and another force PN acts at C so that the system is equivalent to couple. Find the magnitude, direction of P also find the moment of the couple.

6. If three forces acting on a rigidbody is represented magnitue direction and line of action by the three sides of a triangle taken in order show that they once equivalent to a couple of moment represented by twice the area of the triangle.

7. Four forces P, P, Q, Q act along the sides AB, BC, CD, DA of a rhombus ABCD. Find the sum of their moments about the centre O of the rhombus. Prove that their resultant is at a

$$\text{distance } \frac{BD}{2} \left(\frac{P+Q}{P-Q} \right) \text{ from O}$$

Discuss the case when $P = Q$.

4.0 Coplanar forces acting on a rigid body

4.1 Resultant of coplanar forces

In chapter two we discussed the coplanar forces acting on a point. We shall now consider forces acting not all at one point on a rigid body.

Resultant of coplanar forces

We require to find the resultant of number of forces whose magnitude and line of action are given.

The magnitude of the resultant

Resolve the forces in two directions at right angle, add these components separately say **X** and **Y**. The magnitude of the resultant is obtained by $R^2 = X^2 + Y^2$

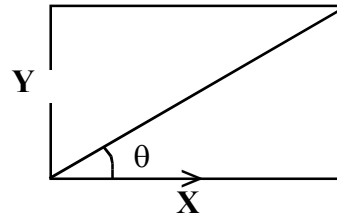
The direction of the resultant

If the angle made with the direction of X is θ .

Then

$$\tan \theta = \frac{Y}{X}$$

$$\theta = \tan^{-1}\left(\frac{Y}{X}\right)$$



To find the position of the line of action.

By taking moments about any point O given line we can find where the line of action of the resultant cuts the line.

Example 1

ABCD is a square of side $2a$. Forces $3P$, $2P$, P , $3P$ Newtons act along the sides AB, CB, CD, AD respectively indicated by the order of letters.

- Find (i) The magnitude and direction
(ii) The line of action of resultant

Resolving Parallel to AB \rightarrow $X = 3P - P$
 $= 2P$

Resolving parallel AD \uparrow $Y = 3P - 2P$
 $= P$

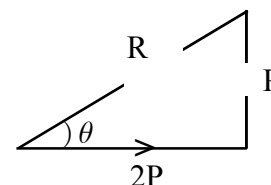
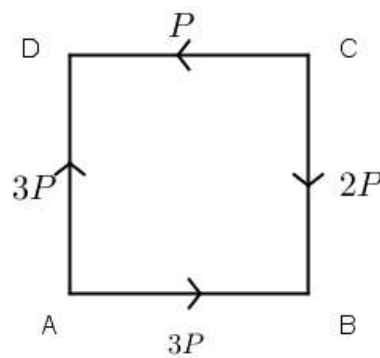
$$R^2 = X^2 + Y^2$$

$$= (2P)^2 + P^2 = 5P^2$$

$$R = P\sqrt{5} \text{ N}$$

$$\tan \theta = \frac{Y}{X} = \frac{P}{2P} = \frac{1}{2}$$

$$\theta = \tan^{-1}\left(\frac{1}{2}\right)$$



Resultant is of magnitude $P\sqrt{5}$ N, makes $\tan^{-1}\left(\frac{1}{2}\right)$ with AB. Let the resultant cuts AB at E where $AE = x$

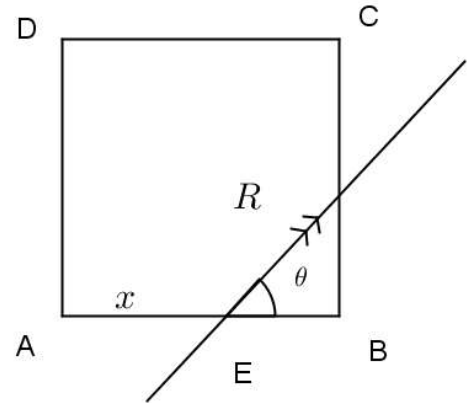
Taking moment about E,

$$\begin{aligned} 3Px + 2P(2a - x) - P \times 2a &= 0 \\ 3x - 2x &= 2a - 4a \\ x &= -2a \end{aligned}$$

OR

taking moments about A

$$\begin{aligned} R \times x \sin \theta &= P \times 2a - 2P \times 2a \\ P\sqrt{5} \times x \times \frac{1}{\sqrt{5}} &= -2Pa \\ x &= -2a \end{aligned}$$



Resultant cuts BA produced at a distance $2a$ from A.

Example 2

ABC is an equilateral triangle of side $2a$. Forces 4, 2, 2, Newtons act along the sides BA, AC, BC in the directions indicated by the order of letters.

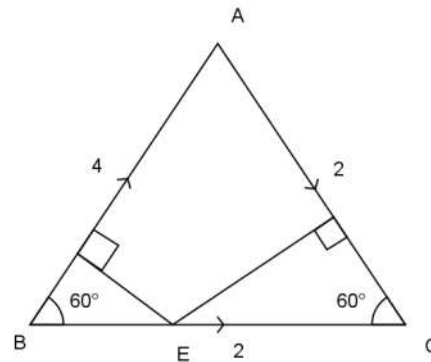
Find the magnitude of the resultant and show that the line of action cuts BC at a distance $\frac{2a}{3}$ from B.

Resolving parallel to BC

$$\begin{aligned} \rightarrow X &= 2 + 2\cos 60 - 4\cos 60 \\ &= 2 + 1 - 2 = 1 \end{aligned}$$

Resolving perpendicular to BC

$$\begin{aligned} \uparrow Y &= 4\sin 60 - 2\sin 60 \\ &= 2 \times \frac{\sqrt{3}}{2} = \sqrt{3} \end{aligned}$$



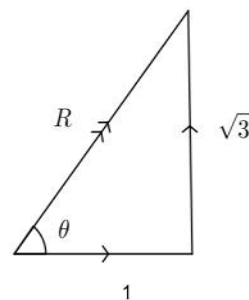
$$R^2 = X^2 + Y^2$$

$$\begin{aligned} &= 1^2 + (\sqrt{3})^2 \\ &= 1 + 3 = 4 \end{aligned}$$

$$R = 2N$$

$$\tan \theta = \frac{Y}{X} = \frac{\sqrt{3}}{1} = \sqrt{3}$$

$$\theta = \tan^{-1}(\sqrt{3}) = 60^\circ$$



Magnitude of resultant $2N$, making 60° with BC

Suppose that the resultant cuts BC at E

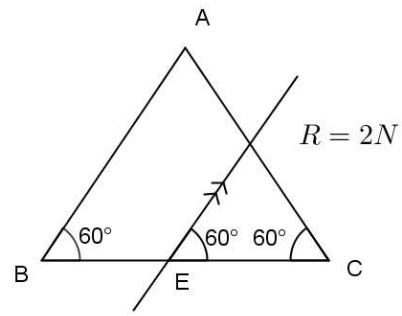
Taking moments about E

$$4 \times x \sin 60 - 2 \times (2a - x) \sin 60 = 0$$

$$4x - 4a + 2x = 0$$

$$6x = 4a$$

$$x = \frac{2}{3} a$$



Example 3

ABCDEF is a regular hexagon of side a metres. Forces of 2, 2, 3, 2 Newton act along the sides AB, CD, ED, EF respectively indicated by the order of letters. Find the magnitude of the resultant and show that it acts through A along AB.

AB and AE are perpendicular to each other.

Resolve the forces parallel to AB

$$\begin{aligned} \rightarrow X &= 2 + 3 - 2\cos 60 - 2\cos 60 \\ &= 3 \end{aligned}$$

Resolving parallel to AE

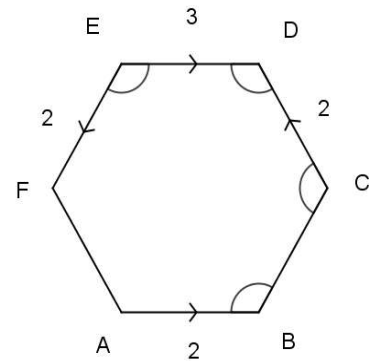
$$\begin{aligned} Y &= 2\sin 60 - 2\sin 60 \\ &= 0 \end{aligned}$$

$$R = 3N \text{ parallel to AB}$$

Taking moment about A

$$\begin{aligned} m \ 2 \times 2a \sin 60 - 3 \times 4a \cos 30 + 2 \times 4a \cos 30 \\ = 0 \end{aligned}$$

It acts through A along AB



Reducing a system of coplanar forces

Any system of coplanar forces acting on a rigid body can, in general, be reduced to a single force acting at an arbitrary point in the plane of the forces together with a couple.

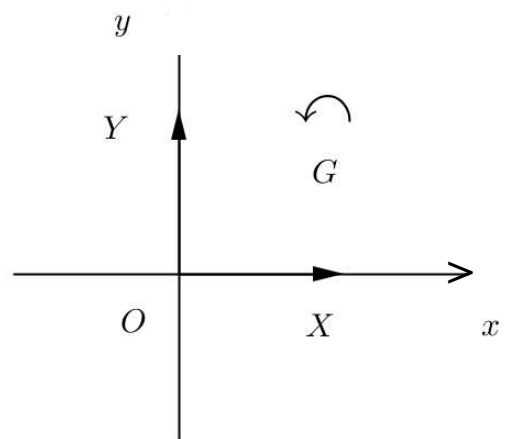
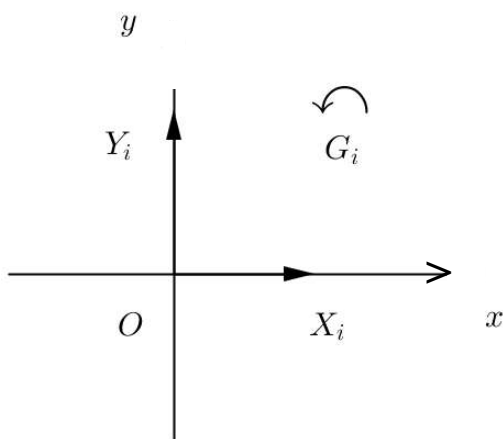
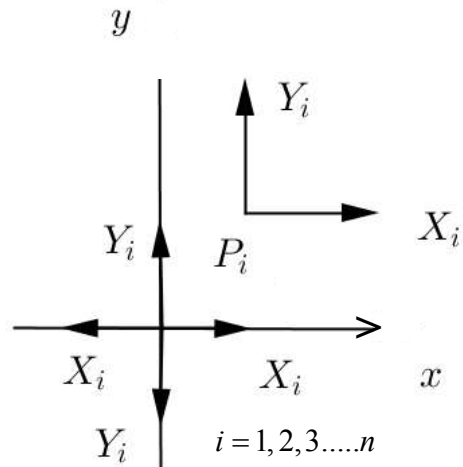
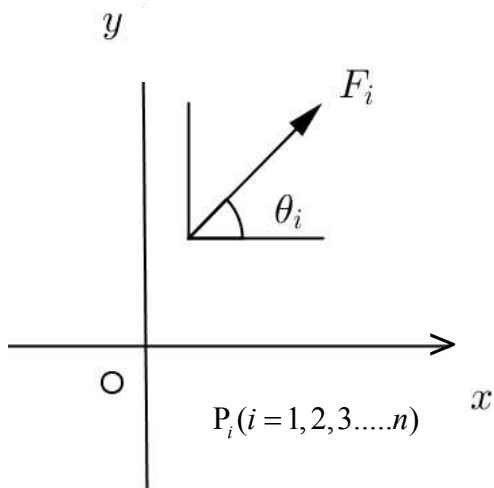
Let the forces $F_i (i = 1, 2, \dots, n)$ act at points $P_i (i = 1, \dots, n)$ in a plane and O be any point in the plane of the forces. Take O as origin of coordinate axes referred to rectangular axes Ox, Oy coordinate of $P_i \equiv (x_i, y_i) (i = 1, 2, \dots, n)$

Let forces $F_i (i = 1, 2, 3, \dots, n)$ makes an angle θ_i with Ox axis

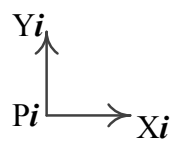
Resolve force F_i into components X_i, Y_i .

Where $X_i = F_i \cos \theta_i, Y_i = F_i \sin \theta_i (i = 1, 2, \dots, n)$

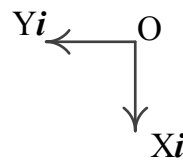
Introduce equal and opposite forces X_i, Y_i at O . This has no effect in the given system of forces,



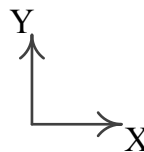
Now at



and at O



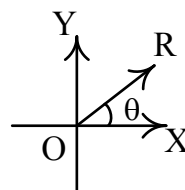
forms a couple of moment G_i and at O the force is



$$G_i \text{ m} = Y_i x_i - X_i y_i$$

$$\text{Let } X = \sum_{i=1}^n X_i$$

$$Y = \sum_{i=1}^n Y_i$$



Then $R^2 = X^2 + Y^2$

$$\tan\theta = \frac{Y}{X}$$

$$\theta = \tan^{-1}\left(\frac{Y}{X}\right)$$

and $G = \sum_{i=1}^n Y_i x_i - X_i y_i$

Note : G is the sum of the moments of all forces in the given system about O , and will depend on the position of O .

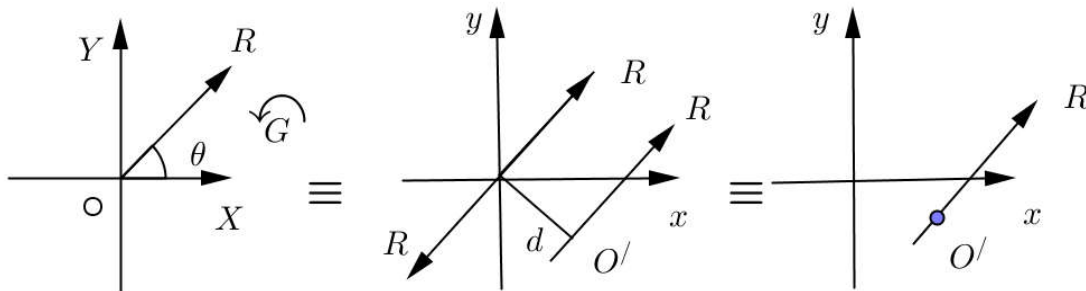
Conditions of equilibrium of a system of coplanar forces

Any system of forces can be reduced to a single force R at any arbitrary point O (origin) together with a couple G in the plane in general.

If,

- i. $R = 0$ and $G = 0$ the system is in equilibrium
- ii. $R \neq 0$, $G = 0$ the system reduces to a single force acting at O
- iii. $R = 0$ and $G \neq 0$ the system reduces to a couple of moment G
- iv. $R \neq 0$ and $G \neq 0$ the system is not in equilibrium and can be reduced to a single force R acting at another point O'

Where $OO' = \frac{G}{R}$ as shown below.



proof :

Replace couple G by equal unlike parallel forces R and R acting at O and O' where distance between the lines of action $d = \frac{G}{R}$

equal and opposite forces balance each. results single force R at O'

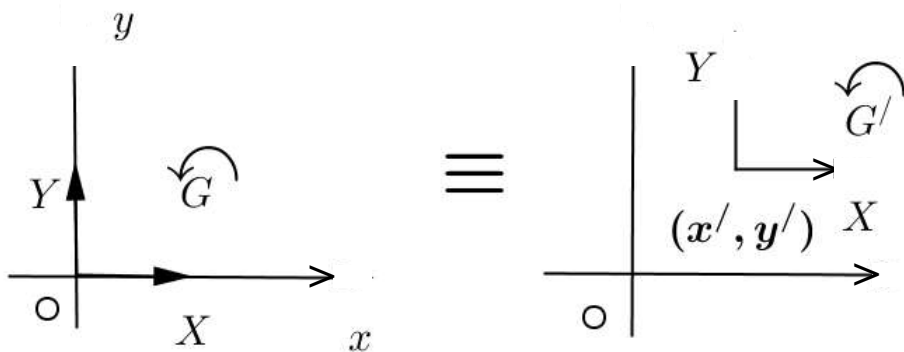
Equation of line of action of the resultant of a system of forces

If a system of forces is not in equilibrium and it can be reduced to a single force R at any point (x', y') together with a couple G'

Then $R^2 = X^2 + Y^2$ and $G' = G + Xy' - Yx'$

Moment about any point in the line of action of the resultant is zero. Let (x, y) be any point on the line of action Then $G + Xy - Yx = 0$;

This is the equation of line of acting of the resultant.



Moment about O = Moment about O

$$G = Yx' - Xy' + G'$$

$$\text{Hence } G' = G + Xy' - Yx'$$

If the resultant passes through (x', y') then $G' = 0$

$$0 = G + Xy' - Yx'$$

and the equation of line of action is

$$0 = G + Xy - Yx$$

4.2 Worked examples

Example 4

Forces 2, 4, 1, 6 N act along the sides AB, CB, CD, AD of a square respectively. Find the magnitude and direction of the resultant.

Prove that the equation of the line of action of the resultant referred to AB and AD as coordinate axis is $2x - y + 3a = 0$

Resolving parallel to AB;

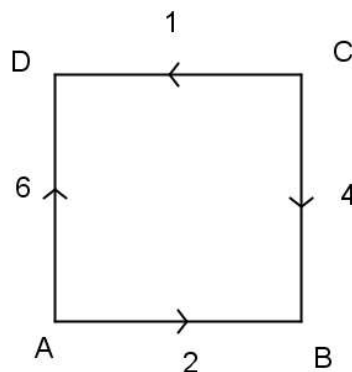
$$\rightarrow X = 2 - 1 = 1$$

Resolving parallel to AD,

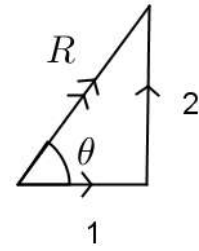
$$\uparrow Y = 6 - 4 = 2$$

$$R^2 = X^2 + Y^2 = 2^2 + 1^2 = 5$$

$$R = \sqrt{5} \text{ N}$$



$$\begin{aligned}\tan\theta &= \frac{X}{Y} \\ &= 2 \\ \theta &= \tan^{-1}(2)\end{aligned}$$



Resultant of magnitude $\sqrt{5}$ N, makes $\tan^{-1}(2)$ with AB.

Taking moments about A,

$$\begin{aligned}m \quad G &= 1 \times a - 4 \times a \\ &= -3a \text{ Nm}\end{aligned}$$

Equation of line of action is

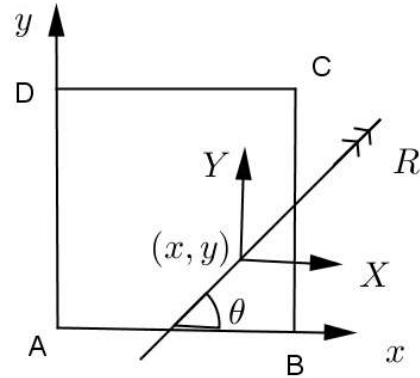
$$\begin{aligned}G + Xy - Yx &= 0 \\ -3a + y - 2x &= 0 \\ 2x - y + 3a &= 0\end{aligned}$$

OR

Moment of the resultant about A

= Algebraic sum of the moments of the forces about A

$$\begin{aligned}G &= Yx - Xy \\ -3a &= 2x - 1y \\ 2x - y + 3a &= 0\end{aligned}$$



Example 5

ABCD is a square of side a . Forces of 5, 4, 3, 2 N act along the sides AB, BC, CD, AD in the direction indicated by the order of letters. Reduce the system to

- a single force at A with a couple.
- a single force at the centre O with a couple.
- Referred to AB and AD as axis find the equation of line of action

Resolving parallel to AB

$$\begin{aligned}\rightarrow X &= 5 - 3 \\ &= 2\end{aligned}$$

Resolving parallel to AD

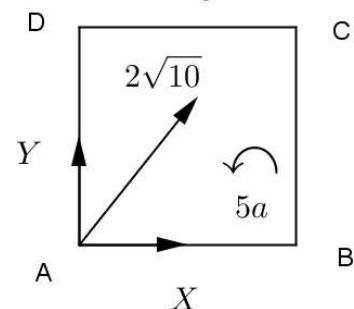
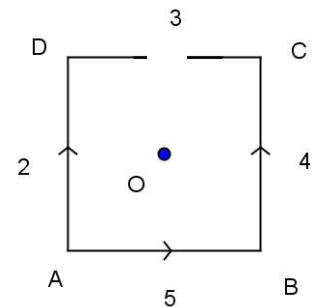
$$\begin{aligned}\uparrow Y &= 4 + 2 \\ &= 6\end{aligned}$$

$$\begin{aligned}R^2 &= X^2 + Y^2 \\ &= 40 \\ R &= 2\sqrt{10}\end{aligned}$$

moment about A

$$\begin{aligned}G &= 4 \times a + 3 \times a \\ &= 7a \text{ Nm}\end{aligned}$$

single force of $2\sqrt{10}$ N with a couple of $7a$ Nm at A



Moment about O,

$$G = 5 \times \frac{a}{2} + 4 \times \frac{a}{2} + 3 \times \frac{a}{2} - 2 \times \frac{a}{2}$$

$$= 5a \text{ Nm}$$

At the centre, $2\sqrt{10}$ N force with a couple $5a$ Nm.

Equation of line of action

$$G + X.y - Y.x = 0$$

$$5a + 2y - 4x = 0$$

$$4x - 2y - 5a = 0$$

Example 6

ABCDEF is a regular hexagon of side $2a$. Forces of 2, 1, 2, 3, 2, 1 N act along the sides AB, BC, CD, ED, EF, AF in the direction indicated by the order of letters respectively

- show that the system can be reduced to a force of magnitude $2\sqrt{3}$ N along AD with a couple. Find the moment of the couple.
- Show that the system can be reduced to a single force and find the equation of its line of action.
- If the line of action cuts FA produced at K find the length of AK.

$$\rightarrow X = 2 + 3 + 1 \cos 60 - 2 \cos 60 - 2 \cos 60 - 1 \cos 60$$

$$= 5 - 2 = 3 \text{ N}$$

$$\uparrow Y = 1 \sin 60 + 1 \sin 60 + 2 \sin 60 - 2 \sin 60$$

$$= \sqrt{3} \text{ N}$$

$$R^2 = X^2 + Y^2$$

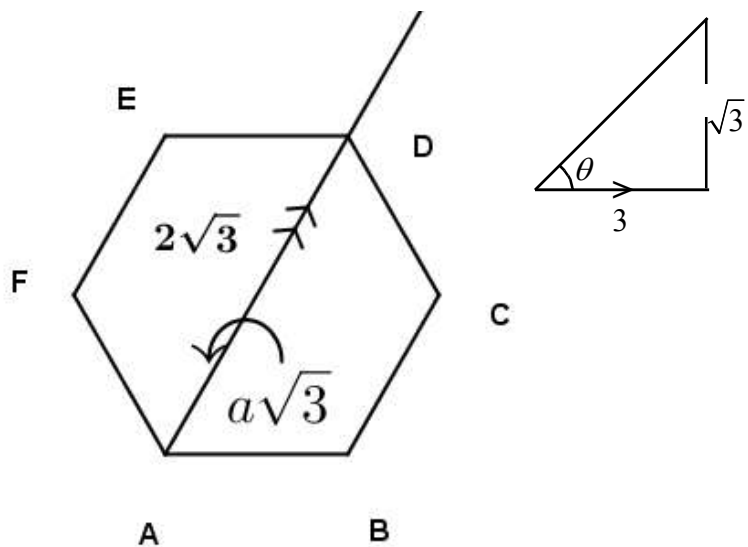
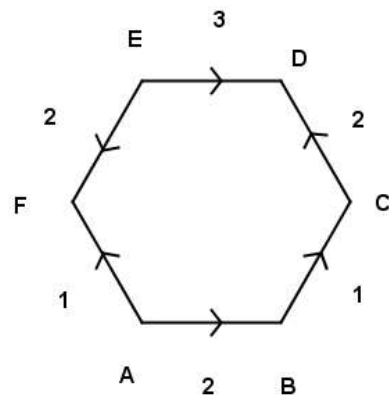
$$= 12$$

$$R = 2\sqrt{3} \text{ N}$$

$$\tan \theta = \frac{X}{Y} = \frac{1}{\sqrt{3}}$$

$$\theta = 30^\circ$$

Resultant is parallel to AD



Sum of moment of the forces about A

$$G = 1 \times 2a \sin 60 + 2 \times 4a \cos 30 + 2 \times 2a \sin 60 - 3 \times 4a \cos 30$$

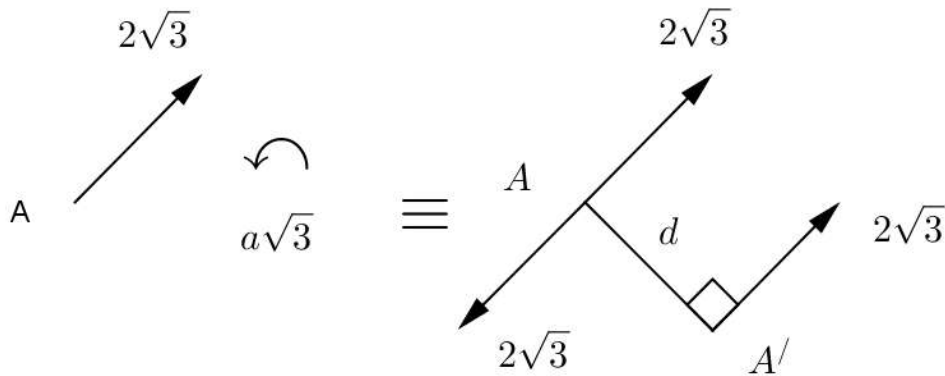
$$= a\sqrt{3} + 4\sqrt{3}a + 2\sqrt{3}a - 6\sqrt{3}a$$

$$= a\sqrt{3} \text{ Nm}$$

system can be reduced to G

force of $2\sqrt{3}$ N along AD with a couple of moment $a\sqrt{3}$ N

moment of the couple $a\sqrt{3}$ Nm



$2\sqrt{3}$ Nm \curvearrowright can be replaced by equal opposite parallel forces of $2\sqrt{3}$ N shown above where

$$AA' = \frac{a\sqrt{3}}{2\sqrt{3}} = \frac{a}{2} \text{ m}$$

The forces at A balance each other,

Therefore it reduces to a single force $2\sqrt{3}$ N at A'

equation of line of action

$$G + Xy - Yx = 0$$

$$a\sqrt{3} + 3y - \sqrt{3}x = 0$$

$$x - \sqrt{3}y - a = 0$$

The line of action cuts AB at H at FA produced at K

$$\text{Let } E = (x_1, 0) \quad x - \sqrt{3}y - a = 0$$

$$y = 0 \quad x_1 = a$$

$$E = (a, 0)$$

$$\sin 60 = \frac{AK}{AE}$$

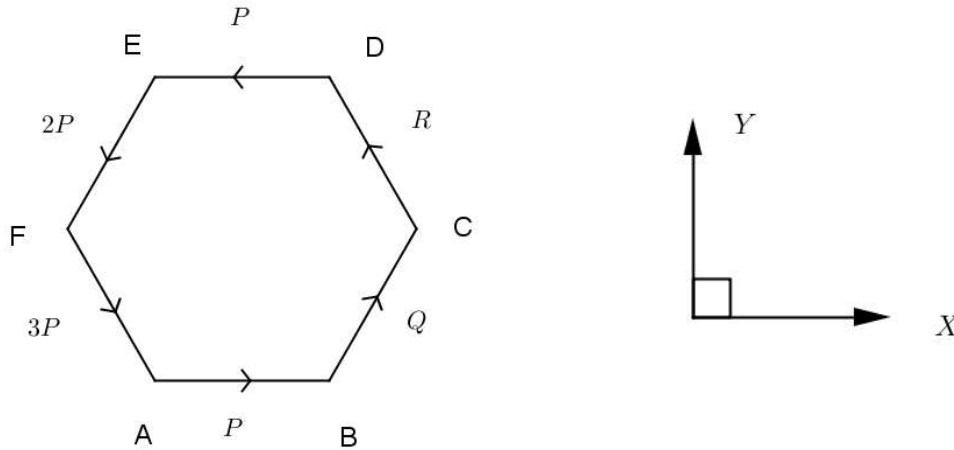
$$AK = AE \sin 60$$

$$= \frac{a\sqrt{3}}{2} \text{ m}$$

Example 7

Forces P, Q, R, P, 2P, 3P N act along the sides AB, BC, CD, DE, EF, FA respectively of a regular hexagon ABCDEF of side 2a metres in the sense indicated by the order of letters.

- i. If the system is equivalent to a couple show that $Q = 2P$ and $R = 3P$ and calculate the moment of the couple.
- ii. If the system is equivalent to a single force along AD find Q and R in terms of P.



System is equivalent to a couple

$$X = 0 \text{ and } Y = 0$$

$$\rightarrow X = Q\cos 60^\circ - R\cos 60^\circ - 2P\cos 60^\circ + 3P\cos 60^\circ = \frac{Q-R+P}{2}$$

$$X = 0; \quad Q - R + P = 0$$

$$R - Q = P \quad \dots\dots\dots(1)$$

$$\uparrow Y = Q\sin 60^\circ + R\sin 60^\circ - 2P\sin 60^\circ - 3P\sin 60^\circ = (Q+R-5P)\frac{\sqrt{3}}{2}$$

$$Y = 0; \quad Q + R = 5P \quad \dots\dots\dots(2)$$

(1), (2) $R = 3P$ and $Q = 2P$

Moment of the couple = sum of the moments about O

$$= (P + Q + R + P + 2P + 3P) \times a\sqrt{3}$$

$$= 12\sqrt{3} aP \text{ Nm}$$

(ii) $X = \frac{Q-R+P}{2}$

$$Y = (Q + R - 5P)\frac{\sqrt{3}}{2}$$

Resultant parallel to AD

$$\theta = 60^\circ$$

$$\tan\theta = \sqrt{3} = \frac{\sqrt{3}(Q+R-5P)}{Q-R+P}$$

$$Q - R + P = Q + R - 5P$$

$$R = 3P$$

Single force along AD

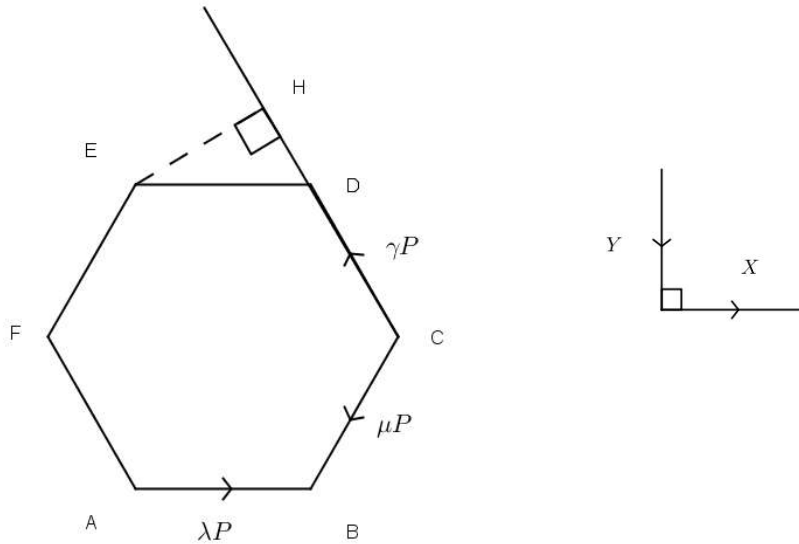
Sum of moments about O is zero,

$$\begin{aligned} (7P + Q + R)a\sqrt{3} &= 0 \\ 10P + Q &= 0 \\ Q &= -10P \end{aligned}$$

Example 8

ABCDEF is a regular hexagon of side a forces of magnitude $\lambda P, \mu P, \gamma P$ act along the sides AB, CB, CD in the sense indicated by the order of letters respectively. The algebraic sum of the moments about verticle D, E, F are $2\sqrt{3} Pa, \frac{3\sqrt{3}}{2} Pa, \frac{\sqrt{3}}{2} Pa$ respectively in anticlockwise direction.

- i. Find the values of λ, μ, γ .
- ii. show that the resultant is a single force, parallel to EC through A of magnitude of $\sqrt{3} PN$.



Moment about D

$$\begin{aligned} m \lambda P \times 2a \cos 30 - \mu P \times a \sin 60 &= 2\sqrt{3} aP \\ 2\lambda \frac{\sqrt{3}}{2} - \mu \frac{\sqrt{3}}{2} &= 2\sqrt{3} \\ 2\lambda - \mu &= 4 \quad \dots\dots\dots(1) \end{aligned}$$

Sum of moments about E

$$\begin{aligned} m \lambda P \times 2a \cos 30 - \mu P \times 2a \cos 30 + \gamma P \times a \sin 60 \\ aP \frac{\sqrt{3}}{2} [2\lambda - 2\mu + \gamma] &= 3a \frac{P\sqrt{3}}{2} \\ 2\lambda - 2\mu + \gamma &= 3 \quad \dots\dots\dots(2) \end{aligned}$$

$$AE = BF = CE = 2a \cos 30 = a\sqrt{3}$$

$$EH = a \sin 60 = a \frac{\sqrt{3}}{2}$$

Sum of moments about F

$$\lambda P a \frac{\sqrt{3}}{2} - \mu P 2a \frac{\sqrt{3}}{2} + \gamma P 2a \frac{\sqrt{3}}{2}$$

$$a p \frac{\sqrt{3}}{2} [\lambda - 2\mu + 2\gamma] = a \frac{P\sqrt{3}}{2}$$

$$\lambda - 2\mu + 2\gamma = 1 \dots\dots\dots (3)$$

(1), (2), (3) $\lambda = 3, \mu = 2, \gamma = 1$

$$\rightarrow X = \lambda P - \mu P \cos 60 - \gamma P \cos 60$$

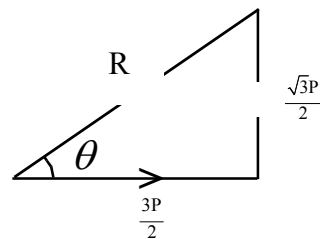
$$= 3P - 2 \frac{P}{2} - \frac{P}{2} = \frac{3P}{2}$$

$$\uparrow Y = \mu P \sin 60 - \gamma P \sin 60 = P \frac{\sqrt{3}}{2}$$

$$R^2 = \left(3 \frac{P}{2}\right)^2 + \left(P \frac{\sqrt{3}}{2}\right)^2 = 3P^2$$

$$R = P\sqrt{3} \text{ N}$$

$$\tan \theta = \frac{\sqrt{3}}{3} = \frac{1}{\sqrt{3}} \quad \theta = 30^\circ$$



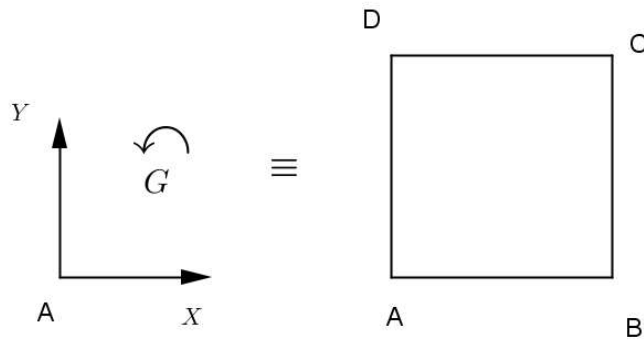
Moment about A m $P \times 2a \cos 30 - 2P \times a \cos 60$
 $= 0$

Resultant is a single force through A of magnitude $P\sqrt{3}$ N as in parallel to EC.

Example 9

ABCD is a rectangle with $AB = 2a, AD = 2a$. The moment of a system of forces in the plane of the rectangle are M_1, M_2, M_3 about points A, B and C respectively

- i. Find the moment of the system about D.
- ii. Determine the magnitude and direction of the resultant of the system.
- iii. Find the equation of line of action of the resultant and if the line of action is perpendicular to BC show that $M_1 = 5M_2 + 4M_3$



Let the system of forces reduced at A as shown

Moment of a system about any point (x, y) in its plane

$$\text{Let } A = (0, 0), \quad B = (2a, 0), \quad C = (2a, a), \quad D = (0, a)$$

Then about A

$$G' = G + Xy - Yx$$

$$M_1 = G + X \cdot 0 - Y \cdot 0$$

$$G = M_1$$

$$\text{about B } M_2 = M_1 + X(0) - Y(2a)$$

$$Y = \frac{M_1 - M_2}{2a}$$

$$\text{about C } -M_3 = M_1 + X(x) - Y(2a)$$

$$-M_3 = M_1 + Xa - (M_1 - M_2)$$

$$X = -\frac{(M_2 + M_3)}{a}$$

moment about $D = (0, a)$

$$G' = M_1 - \frac{(M_2 + M_3)}{a} \times a - \left(\frac{M_1 - M_2}{2a} \right) \times 0$$

$$= (M_1 - M_2 - M_3)$$

$$R^2 = X^2 + Y^2$$

$$= \left(\frac{M_2 + M_3}{a} \right)^2 + \left(\frac{M_1 - M_2}{2a} \right)^2$$

$$= \frac{4(M_2 + M_3)^2 + (M_1 - M_2)^2}{4a^2}$$

$$R = \frac{1}{2a} \left[(M_2 + M_3)^2 + (M_1 - M_2)^2 \right]^{\frac{1}{2}}$$

$$\begin{aligned}\tan\theta &= \frac{Y}{X} \\ &= \left(\frac{M_1 - M_2}{2a}\right) \times \left(\frac{-a}{M_2 + M_3}\right) = \frac{1}{2} \left(\frac{M_2 - M_1}{M_2 + M_3}\right)\end{aligned}$$

$$\theta = \tan^{-1} \left[\frac{M_2 - M_1}{2(M_2 + M_3)} \right]$$

Equation of line of action

$$G + Xy - Yx = 0$$

$$M_1 - \frac{(M_2 + M_3)}{a}y - \left(\frac{M_1 - M_2}{2a}\right)x = 0$$

$$(M_1 - M_2)x + 2(M_2 + M_3)y - 2aM_1 = 0$$

$$\text{Gradient of the line} = \frac{(M_2 - M_1)}{2(M_2 + M_3)}$$

$$\text{Gradient of AC} = \frac{1}{2}$$

$$\frac{(M_2 - M_1)}{2(M_2 + M_3)} \times \frac{1}{2} = -1$$

$$\begin{aligned}M_2 - M_1 &= -4M_2 - 4M_3 \\ 5M_2 + 4M_3 &= M_1\end{aligned}$$

Example 10

A system of forces F_i ($i = 1, 2, \dots, n$) act at points P_i whose coordinates (x_i, y_i) related to rectangular axis Ox, Oy . Each force of the system makes an angle θ with Ox .

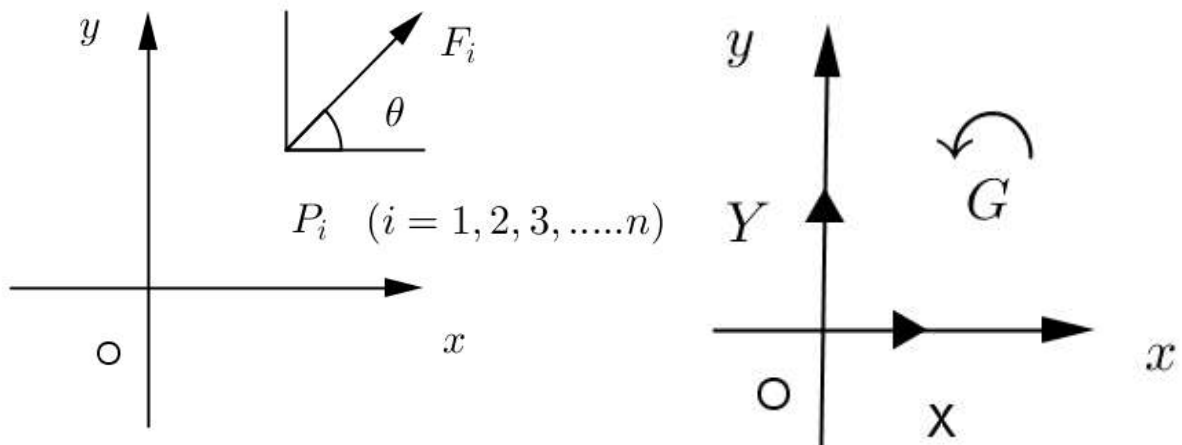
- i. Reduce the system as a single force at O together with a couple.
- ii. Write down the equation of the line of action of the resultant
- iii. Reduce as θ varies, the corresponding resultant of the system passes through a fixed point in the plane and find its coordinate.

$$X_i = F_i \cos \theta \quad i = (1, 2, \dots, n)$$

$$X = \sum_{i=1}^n X_i = \cos \theta \sum_{i=1}^n F_i$$

$$Y_i = F_i \sin \theta \quad i = (1, 2, \dots, n)$$

$$Y = \sum_{i=1}^n Y_i = \sin \theta \sum_{i=1}^n F_i$$



Moment of a Force F_i about O is

$$G_i = Y_i x_i - X_i y_i$$

Sum of the moments

$$\begin{aligned}
 G &= \sum_{i=1}^n G_i \\
 &= \sum_{i=1}^n (Y_i x_i - X_i y_i) \\
 &= \sum_{i=1}^n F_i x_i - \sum_{i=1}^n F_i y_i \\
 R^2 &= X^2 + Y^2 \\
 &= \cos^2 \theta \left(\sum_{i=1}^n F_i \right)^2 + \sin^2 \theta \left(\sum_{i=1}^n F_i \right)^2 \\
 &= \left(\sum_{i=1}^n F_i \right)^2 \\
 R &= \sum_{i=1}^n F_i
 \end{aligned}$$

equation of line of action

$$G + Xy - Yx = 0$$

$$\sin \theta \sum_{i=1}^n F_i x_i - \cos \theta \sum_{i=1}^n F_i y_i + y \cos \theta \sum_{i=1}^n F_i - x \sin \theta \sum_{i=1}^n F_i = 0$$

$$\sin \theta \left(\sum_{i=1}^n F_i x_i - x \sum_{i=1}^n F_i \right) - \cos \theta \left(\sum_{i=1}^n F_i y_i - y \sum_{i=1}^n F_i \right) = 0$$

$$x \sin \theta \sum_{i=1}^n F_i - y \cos \theta \sum_{i=1}^n F_i + \cos \theta \sum_{i=1}^n F_i y_i - \sin \theta \sum_{i=1}^n F_i x_i = 0$$

This is a variable line depends on θ .

as θ varies,

The line passes through a point

where
$$x = \frac{\sum_{i=1}^n F_i x_i}{\sum_{i=1}^n F_i} \quad y = \frac{\sum_{i=1}^n F_i y_i}{\sum_{i=1}^n F_i}$$

independent of θ .

Edit of the fixed point $\left(\frac{\sum_{i=1}^n F_i x_i}{\sum_{i=1}^n F_i}, \frac{\sum_{i=1}^n F_i y_i}{\sum_{i=1}^n F_i} \right)$ OR

$$\sin\theta \left(\sum_{i=1}^n F_i x_i - x \sum_{i=1}^n F_i \right) - \cos\theta \left(\sum_{i=1}^n F_i y_i - y \sum_{i=1}^n F_i \right) = 0$$

is a straight line passing through the point of intersection of two lines

$$\sum_{i=1}^n F_i x_i - x \sum_{i=1}^n F_i = 0 \quad \text{and} \quad \sum_{i=1}^n F_i y_i - y \sum_{i=1}^n F_i = 0$$

for all values of θ

Hence point of intersection is $\left(\frac{\sum_{i=1}^n F_i x_i}{\sum_{i=1}^n F_i}, \frac{\sum_{i=1}^n F_i y_i}{\sum_{i=1}^n F_i} \right)$

Example 11

Forces P, 7P, 8P, 7P, 3P newtons act along the sides AB, CB, CD, ED and FE respectively of a regular hexagon ABCDEF of side a meters in the direction indicated by the order of letters.

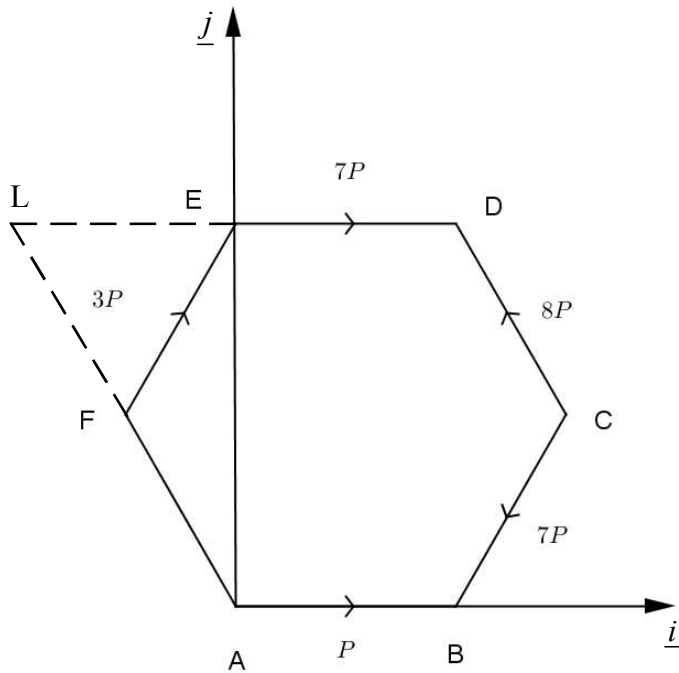
Taking \hat{i} and \hat{j} be the unit vectors along the directions \overline{AB} and \overline{AD} respectively express each force in terms of \hat{i}, \hat{j} and P. Show that the given system is equivalent to a single force

$$\underline{R} = 2P(\hat{i} + \sqrt{3}\hat{j}) \text{ parallel to } \overline{BC} .$$

What is the magnitude of \underline{R}

Show further that the line of action of the resultant passes through the common point of DE and AF (both produced)

If the system is equivalent to a force R acting through A together with a couple find the moment of this couple.



Forces are,

$P\mathbf{i}$ along \overrightarrow{AB}

$7P\left(-\frac{1}{2}\mathbf{i}-\frac{\sqrt{3}}{2}\mathbf{j}\right)$ along \overrightarrow{CE}

$8P\left(-\frac{1}{2}\mathbf{i}+\frac{\sqrt{3}}{2}\mathbf{j}\right)$ along \overrightarrow{CD}

$7P\mathbf{i}$ along \overrightarrow{ED}

$3P\left(\frac{1}{2}\mathbf{i}-\frac{\sqrt{3}}{2}\mathbf{j}\right)$ along \overrightarrow{FE}

$$\text{Resultant } \mathbf{R} = \left(1 - \frac{7}{2} - \frac{8}{2} + 7 + \frac{3}{2}\right) P\mathbf{i} + \left(-\frac{7\sqrt{3}}{2} + \frac{8\sqrt{3}}{2} + \frac{3\sqrt{3}}{2}\right) P\mathbf{j}$$

$$= 2P\mathbf{i} + 2\sqrt{3}P\mathbf{j}$$

$$= 2P(\mathbf{i} + \sqrt{3}\mathbf{j}) \text{ is single resultant force}$$

$$\overrightarrow{BC} = \frac{a}{2}\mathbf{i} + \frac{a\sqrt{3}}{2}\mathbf{j}$$

$$= \frac{a}{2}(\mathbf{i} + \sqrt{3}\mathbf{j})$$

$\therefore \mathbf{R}$ is parallel to \overrightarrow{BC}

$$\begin{aligned} |\underline{R}| &= \sqrt{(2P)^2 + (2\sqrt{3}P)^2} \\ &= 4P \end{aligned}$$

Taking moment about L,

$$P \times 2a \cos 30 - 7P \times 3a \cos 30 + 8P \times 2a \sin 60 + 3P \times a \sin 60$$

$$21P a \sin 60 - 21P a \cos 30 = 0$$

resultant pass through the intersection point of DE, AF produced.

4.3 Exercises

1. Forces 1, 3, 5, 7, $9\sqrt{2}$ act along the sides AB, BC, CD, DA and the diagonal BD of a square ABCD of side a in the sense indicated by the order of letters. Taking AB and AD as axes of x and y respectively find
 - i. Magnitude and direction of the resultant
 - ii. Equation of the line of action of the resultant
 - iii. The point in which the resultant cuts AB.

2. ABCDEF is a regular hexagon of side a . Forces 1, 3, 2, 4 N act along AB, BE, ED and DA respectively indicated by the order of letters. Taking AB and AD as x and y axes respectively, find
 - i. magnitude and direction of resultant
 - ii. equation of line of action.

3. Forces of magnitude $F, 2F, 3F, 4F, 5F, 6F$ act along the sides AB, BC, CD, DE, DF, FA of a regular hexagon of side a taken in order. Show that
 - i. They are equivalent to a single force $6F$ acting parallel to one of the given forces.
 - ii. The distance of the line of action of that force and of the resultant from the centre of the hexagon is in the ratio 2 : 7

4. Forces 4, 3, 3 N act along the sides AB, BC, CA of an equilateral triangle ABC of side 0.6 m indicated by the order of letters. Find
 - i. The magnitude and direction of the resultant
 - ii. The perpendicular distance of the line of action from C
 - iii. If an additional force F is introduced at C in the plane of ABD, the system is now equivalent to a couple. Find the moment of the couple and magnitude and direction of the force introduced.

5. The coordinates of the points O, A, B, C are (0, 0), (3, 0), (3, 4) and (0, 4) respectively. Forces of magnitude 7, 6, 2, 9, 5 N act along CA, AB, BC, CO, OB indicated by the order of letters and a couple of moment 16 units acts in the plane in the sense OCBA. Reduce the given system to a force and a couple at O. Show that the system is equivalent to a single force acting along the line $3x - 4y - 5 = 0$.
6. ABCDEF is a regular hexagon of centre O and length of a side a metres. Five forces P, 2P, 3P, 4P, 5P Newton act along the sides AB, BC, CD, DE, EF respectively in the direction indicated by the order of letters. Three new forces Q, R, S newtons acting along AF, FO, OA respectively are added to the system. Find the values of Q, R, S in terms of P.
- if the whole system is in equilibrium
 - equivalent to a couple of moment $Pa\sqrt{3}$ Nm in the sense ABC.
7. The points A, B, C, D, E, F are vertices of a regular hexagon of side $2a$ m in anticlockwise sense. Forces of magnitudes P, 2P, P, mP , nP and 2P newtons act along the sides AB, CB, DC, DE, FE and FA respectively in the sense indicated by the order of letters.
- If the system reduces to a single force acting along DA find the values of m and n .
 - A clockwise couple of magnitude $2\sqrt{3} Pa$ Nm in the place of the hexagon is added to the single show that the new system reduces to a single force and find the point of its line of action with AB produced if necessary.
8. Let ABCD be a square of side a metres. Forces of magnitudes 4, $6\sqrt{2}$, 8, 10, X and Y newtons act along AD, CD, AC, BD, AB and CB respectively, in directions indicated by the order of letters. The system reduces to a single resultant acting along \overrightarrow{OE} , where O and E are the midpoints of AC and CD respectively. Find the values of X and Y, and show that the magnitude of the resultant is $4K$ newtons, where $K = 2 - \sqrt{2}$.
Let F be the point such that OAFD is a square. Find the two forces, one along \overrightarrow{AD} and the other through the point F, which are equivalent to the above system. A couple of moment $6ka$ newton metres acting in the sense ABCD, in the plane of the forces, is added to the original system. Find the line of action of the new system.
9. ABC is an equilateral triangle; O is the centre and R is the radius of the circumcircle of the triangle ABC. A system consists of six forces of magnitudes L, L, M, M and N, N acting along BC, OA, CA, OB, AF and OC respectively in the sense indicated by the order of the letters and a non-zero couple of moment $\lambda R(L + M + N)$ acting in the plane of the triangle ABC, in the sense ACB.
Show that if the system reduces to
- a single force, then $L^2 + M^2 + N^2 > LM + MN + NL$
 - a single couple, then $L = M = N$, $\lambda \neq \frac{1}{2}$.
- State a set of necessary and sufficient conditions for this system to be in equilibrium.

10. ABCD is a square of side m , E is on AB such that $AE = 3m$. The forces λP , μP , νP , $2P$, $10P$ and $2\sqrt{2}P$ Newtons act along the directions BA, BC, CD, AD, DE and DB respectively as indicated by the order of the letters.
- When the system is in equilibrium show that $\lambda = \mu = 6$ and $\nu = 4$
 - If $\nu \neq 4$ and $\lambda = \mu = 6$ then show that the system reduces to a single force and find its magnitude direction and line of action.
 - If $\nu = 2$ and $\lambda = \mu = 6$ then find the magnitude, direction and the line of action of the force that should be added to the system so that the system reduces to a couple of moment $80 Nm$.

11. Forces P , $7P$, $8P$, $7P$, $3P$ newtons act along the sides AB, CB, CD, ED, FE respectively of a regular hexagon ABCDEF of side a meters in the direction indicated by the order of letters. Taking \underline{i} and \underline{j} to be unit vectors in the direction \overrightarrow{AB} and \overrightarrow{AE} respectively. Express each force in terms of \underline{i} , \underline{j} and P .

Show that the given system is equivalent to a single resultant force $\underline{R} = 2P(\underline{i} + \sqrt{3}\underline{j})$ parallel to \overrightarrow{BC} .

What is the magnitude of \underline{R} ?

Show that the line of action of the resultant passes through the common points of DE and AF (both produced).

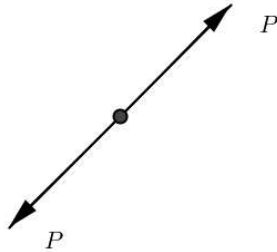
If the system is equivalent to a force \underline{R} acting through the vertex A together with a couple, find the moment of the couple, in magnitude and sense.

12. The coordinates of the points A, B and C with respect to a rectangular cartesian axes Ox and Oy are $(\sqrt{3}, 0)$, $(0, -1)$ and $\left(2\frac{\sqrt{3}}{3}, 1\right)$ respectively. Forces of magnitude $6P$, $4P$, $2P$ and $2\sqrt{3}P$ newtons act along OA, BC, CA and BO respectively in the directions indicated by the order of letters. Find the magnitude and direction of the resultant of these forces. Find the point at which the line of action of the resultant of these forces. Find the point at which the line of action of the resultant cuts the y axis. Hence find the equation of the line of action of the resultant.

Another force of magnitude $6\sqrt{3}P$ newtons is introduced to the system along \overrightarrow{AB} show that the system is reduced to a couple of magnitude $10P$ newton metre.

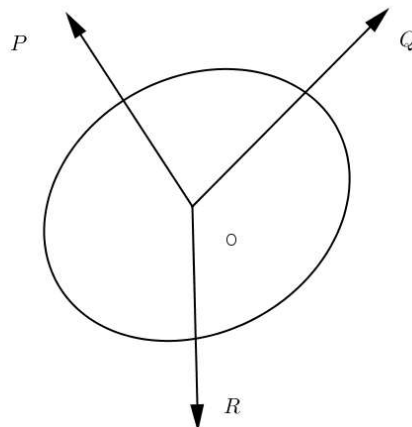
4.4 Equilibrium of a rigid body under the action of coplanar forces

(1) Under the action of two forces



If the two forces are equal in magnitude acting in opposite direction along the same line then the body will be in equilibrium.

(2) Under the action of three Forces



We have two cases to consider.

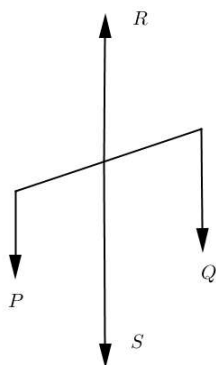
- (i) All three forces are not parallel
- (ii) They all parallel

In (i) they all should meet at one point and resultant of any two forces should equal and opposite direction to the third one.

Proof:

Let P, Q, R be the three forces acting on a rigid body and P, Q meet at a point O . Then P, Q have a resultant passing through O . Now we have two forces, therefore for equilibrium R should pass through O and equal, opposite direction with the resultant of P and Q .

In (ii) all P,Q,R are parallel.



Let P,Q are like parallel forces have a resultant S parallel to P or Q. Now we have two parallel forces S and R. For equilibrium S and R to be equal, dislike parallel and to be in the same line of action , otherwise they have a resultant or forms a couple.

When a rigid body is in equilibrium under the action of three coplanar forces the following results can be used

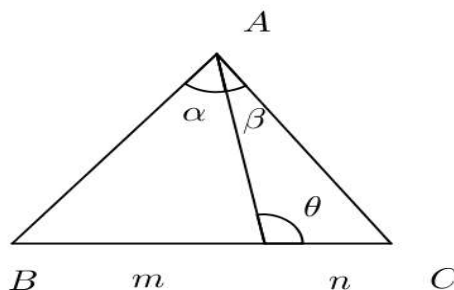
- i Lami's Theorem
- ii Triangle of Forces
- iii The sum of resolved components along two perpendicular direction are zero

Also the following Trigonometric theorem is useful in dealing with equilibrium problems.

Theorem :

In a triangle ABC let D is a point on BC such that $BD : DC = m : n$ and $\hat{B}AD = \alpha$, $\hat{C}AD = \beta$, $\hat{A}DC = \theta$ then

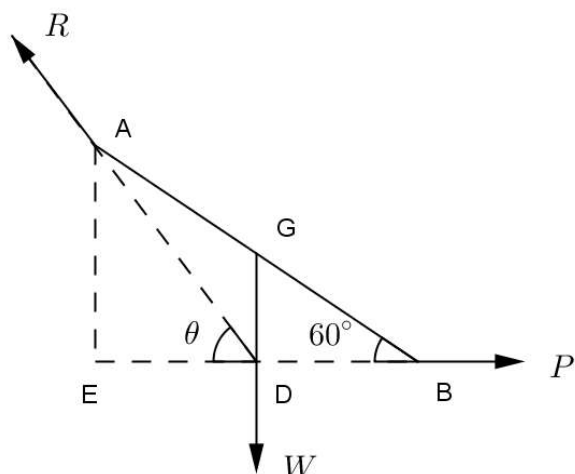
- (i) $(m + n) \cot \theta = m \cot \alpha - n \cot \beta$
- (ii) $(m + n) \cot \theta = n \cot B - m \cot C$



4.5 Worked examples

Example 1:

A heavy uniform rod AB is hinged at A to a fixed point and rests in a position inclined 60 to the horizontal being acted upon by a horizontal force P applied at the lower end B. Find the magnitude of P and the reaction at hinge.



The forces acting are

- (i) Weight W of the rod acting vertically through the mid point of the rod.
- (ii) The horizontal force P at B.
- (iii) The reaction R at hinge A.

The rod is in equilibrium of under the action of three forces, therefore must meet at one point. say D

Let $AB = 2a$, $\angle ADE = \theta$

$$AE = 2a \sin 60^\circ = \sqrt{3}a$$

Method (i) Using Lami's Theorem

$$\frac{P}{\sin(90 + \theta)} = \frac{R}{\sin 90^\circ} = \frac{W}{\sin(180 - \theta)}$$

$$\frac{P}{\cos \theta} = R = \frac{W}{\sin \theta}$$

$$P = W \cot \theta$$

$$R = W \operatorname{cosec} \theta$$

$$P = \frac{W}{2\sqrt{3}} = \frac{\sqrt{3}W}{6} N$$

$$R = W \sqrt{\frac{13}{12}} N$$

$$ED = \frac{1}{2} \times 2a \cos 60^\circ = \frac{a}{2}$$

$$\tan \theta = \frac{AE}{ED} = 2\sqrt{3}$$

$$\operatorname{cosec} \theta = \sqrt{1 + \cot^2 \theta}$$

$$\operatorname{cosec} \theta = \sqrt{1 + \frac{1}{12}} = \sqrt{\frac{13}{12}}$$

Method (ii)

In triangle AED, AE is parallel to W, and ED, DA can represent P and R.

ie, $\triangle AED$ is the triangle of forces

$$R \rightarrow DA$$

$$W \rightarrow AE$$

$$P \rightarrow ED$$

$$\text{then } \frac{P}{ED} = \frac{W}{AE} = \frac{R}{DA}$$

$$\frac{P}{\frac{a}{2}} = \frac{W}{\sqrt{3}a} = \frac{R}{\frac{\sqrt{13}a}{2}}$$

$$AD = \sqrt{3a^2 + \frac{a^2}{4}} = \frac{\sqrt{13}}{2} a$$

$$P = \frac{W}{2\sqrt{3}} \quad \text{and} \quad R = W \sqrt{\frac{13}{12}}$$

Method (iii) Resolving Forces

Resolving horizontally \rightarrow

$$P - R \cos \theta = 0$$

$$P = R \cos \theta$$

Resolving vertically \uparrow

$$R \sin \theta - W = 0$$

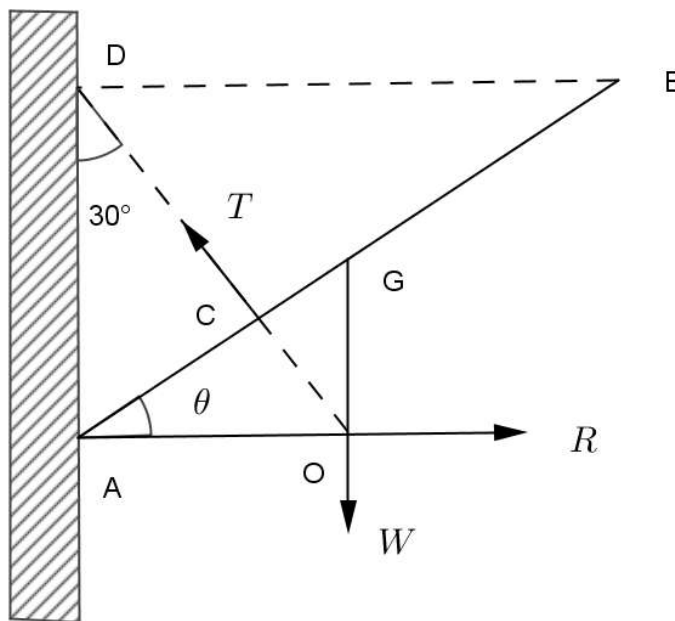
$$R = \frac{W}{\sin \theta} = W \sqrt{\frac{13}{12}} N$$

$$P = W \cot \theta = \frac{W}{2\sqrt{3}} N$$

Example 2:

A uniform rod ABC of weight W is supported with B being uppermost, with its end A against a smooth vertical wall AD by means of a string CD, DB being horizontal and CD is inclined to the wall at an angle of 30° . Find

- i The tension in the string
- ii Reaction of the wall
- iii The inclination of the rod
- iv Prove that $AC = \frac{1}{3}AB$



The forces acting are ,

- i weight W , vertically downward through G , $AG = a$
- ii Reaction R at A , horizontal force R
- iii Tension in the string T

Since the rod is in equilibrium three forces should meet at one point is O .

Let θ be the inclination of the rod to horizontal

Resolving horizontally \rightarrow

$$T \sin 30^\circ - R = 0$$

$$R = \frac{T}{2}$$

Resolving vertically \uparrow

$$T \cos 30 - W = 0$$

$$T = \frac{2W}{\sqrt{3}} = \frac{2\sqrt{3}}{3}W \Rightarrow R = \frac{W}{\sqrt{3}} = \frac{\sqrt{3}W}{3}$$

For equilibrium of AB, Taking moment about D

$$R \times AD - W \times AO = 0$$

$$\frac{W}{\sqrt{3}} \times 2a \sin \theta = W \times a \cos \theta \Rightarrow \tan \theta = \frac{\sqrt{3}}{2} \Rightarrow \theta = \tan^{-1} \left(\frac{\sqrt{3}}{2} \right)$$

In triangle ACD using sin rule

$$\frac{AC}{\sin 30} = \frac{AD}{\sin(120 - \theta)}$$

$$\frac{AC}{\frac{1}{2}} = \frac{2a \sin \theta}{\cos(30 - \theta)}$$

$$AC = \frac{a \sin \theta}{\cos 30 \cos \theta + \sin 30 \sin \theta} = \frac{a}{\cos 30 \cot \theta + \sin 30}$$

$$= \frac{2a}{\sqrt{3} \times \frac{2}{\sqrt{3}} + 1} = \frac{2a}{3}$$

$$AC = \frac{1}{3} AB$$

Method 2 : By Lami's Theorem

$$\frac{T}{\sin 90} = \frac{W}{\sin 180} = \frac{R}{\sin 150}$$

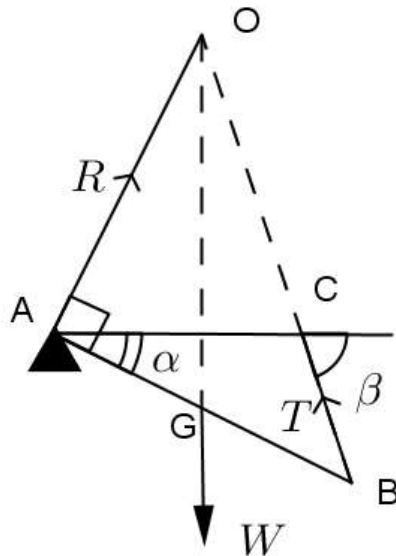
$$T = \frac{W}{\cos 30} \quad R = \frac{\cos 60}{\cos 30}$$

$$T = \frac{2W}{\sqrt{3}} N \quad R = \frac{W}{\sqrt{3}} N$$

Example 3:

A uniform rod AB is in equilibrium at an angle α with the horizontal with its upper end A resting against a smooth peg and its lower end B attached to a light cord, which is pastned to a point C on the same level as A. Prove that the angle β at which the cord is inclined to the horizontal is

given by the equation $\tan \beta = 2 \tan \alpha + \cot \alpha$ and $AC = \frac{AB \sec \alpha}{1 + 2 \tan^2 \alpha}$



The forces acting,

- i Weight W
- ii Tension in the cord T
- iii Reaction at Peg R , perpendicular to the rod

The rod is in equilibrium under the action of three forces, they meet at point O

In triangle AOC , using cot Rule

$$(AG + GB) \cot \angle OGB = GB \cot 90 - AG \cot \angle ABO$$

$$(1+1) \cot(90 + \alpha) = 1 \times \cot 90 - 1 \times \cot(\beta - \alpha)$$

$$2 \tan \alpha = \cot(\beta - \alpha)$$

$$2 \tan \alpha = \frac{1 + \tan \beta \tan \alpha}{\tan \beta - \tan \alpha}$$

$$\tan \beta (2 \tan \alpha - \tan \alpha) = 1 + 2 \tan^2 \alpha$$

$$\tan \beta = \frac{1 + 2 \tan^2 \alpha}{\tan \alpha}$$

$$\tan \beta = \cot \alpha + 2 \tan \alpha$$

$$AG : GB = 1 : 1$$

$$\angle OBA = \beta - \alpha$$

$$\angle OAB = 90^\circ$$

$$\angle OGB = 90^\circ + \alpha$$

Using sin Rule, in triangle ABC

$$\frac{AC}{\sin(\beta - \alpha)} = \frac{AB}{\sin(180 - \beta)}$$

$$AC = \frac{AB \sin(\beta - \alpha)}{\sin \beta}$$

$$AC = \frac{AB}{\sin \beta} [\sin \beta \cos \alpha - \cos \beta \sin \alpha]$$

$$AC = AB [\cos \alpha - \cot \beta \sin \alpha]$$

$$= AB \left[\cos \alpha - \frac{\tan \alpha}{1 + 2 \tan^2 \alpha} \sin \alpha \right]$$

$$= \frac{AB}{1 + 2 \tan^2 \alpha} \left[\cos \alpha + \frac{2 \sin^2 \alpha}{\cos \alpha} - \frac{\sin^2 \alpha}{\cos \alpha} \right]$$

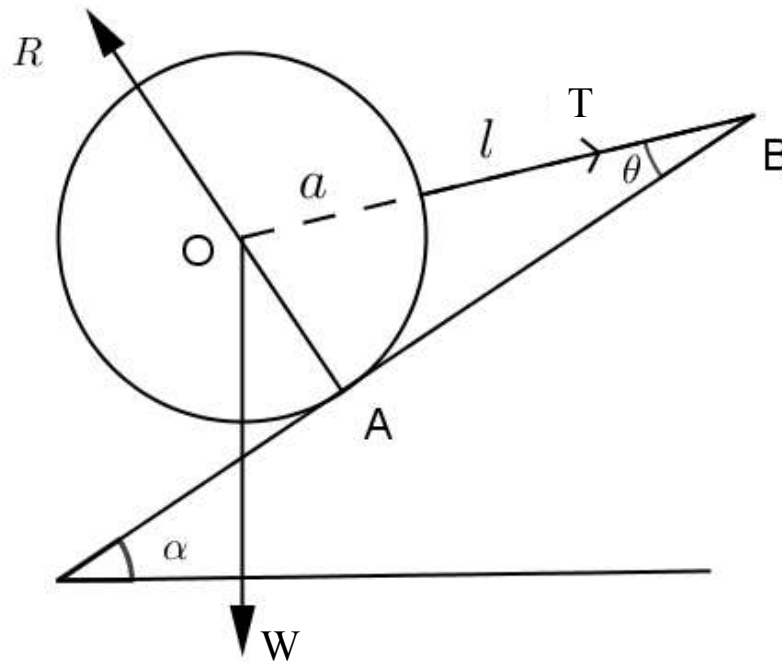
$$= \frac{AB}{1 + 2 \tan^2 \alpha} \left[\frac{\cos^2 \alpha + \sin^2 \alpha}{\cos \alpha} \right]$$

$$AC = \frac{AB \sec \alpha}{1 + 2 \tan^2 \alpha}$$

Example 4:

A sphere of radius a and weight W rests on a smooth inclined plane supported by a string of length l with one end attached to a point on the surface of the sphere and other end fastened to a point on the plane. If the inclination of the plane to the horizontal is α , prove that the tension

in the string is $\frac{W(a+l)\sin\alpha}{\sqrt{l^2+2al}}$



The forces acting are

- i Weight W of the sphere, vertically downward through its centre O
- ii Reaction R of the plane perpendicular to the plane, pass through centre O
- iii Tension in the string T

The sphere is in equilibrium under three forces tension in the string should pass through O
In triangle AOB

$$OB = a + l$$

$$OA = a$$

$$AB^2 = (a + l)^2 - a^2 = l^2 + 2al$$

$$AB = \sqrt{l^2 + 2al}$$

Method 1

Resolving parallel to the plane

$$OT \cos \theta - W \cos(90 - \alpha) = 0$$

$$T = \frac{W \sin \alpha}{\cos \theta}$$

$$= W \sin \alpha \cdot \frac{(a+l)}{\sqrt{l^2+2al}} = \frac{W(a+l)\sin\alpha}{\sqrt{l^2+2al}}$$

$$\cos \theta = \frac{AB}{OB}$$

$$\cos \theta = \frac{\sqrt{l^2+2al}}{a+l}$$

Method 2 : (Lami's Theorem)

$$\frac{R}{\sin(90 + \alpha - \theta)} = \frac{W}{\sin(90 + \theta)} = \frac{T}{\sin \theta}$$

$$T = \frac{W \sin \alpha}{\cos \theta} = \frac{W(a+l) \sin \alpha}{\sqrt{l^2 + 2al}}$$

$$R = \frac{W}{\cos \theta} \cdot \cos(\alpha - \theta)$$

$$= W(\cos \alpha + \sin \alpha \tan \theta)$$

$$= W \left(\cos \alpha + \sin \alpha \cdot \frac{a}{\sqrt{l^2 + 2al}} \right)$$

Example 5

A rod of weight W whose centre of gravity divides its length in the ratio 2:1 lies in equilibrium inside a smooth hollow sphere. If the rod subtends an angle 2α at the centre of the sphere and

makes angle θ with the horizontal, prove that $\tan \theta = \frac{1}{3} \tan \alpha$. Also find the reactions at the end of the rod in terms of W and α

The forces acting are

- (i) Weight of the rod W acting through centre O
- (ii) Reactions at the ends A and B pass through the centre O

$$\angle OCA = 90 - \theta$$

$$\angle OAB = \angle OBA = 90 - \alpha$$

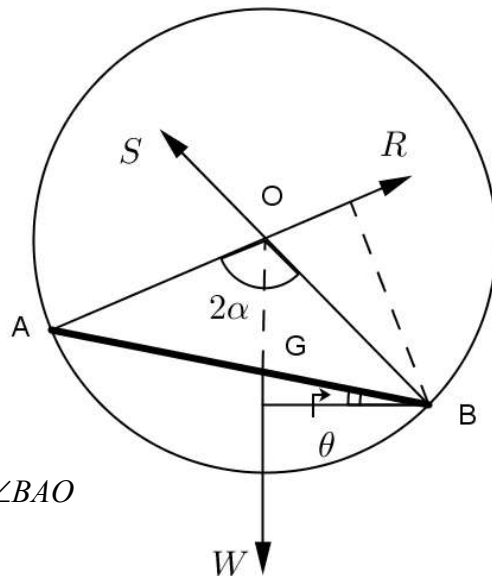
Using cot rule in triangle AOB

$$(BC + CA) \cot(90 - \theta) = CA \cdot \cot \angle ABO - BC \cdot \cot \angle BAO$$

$$3 \cot(90 - \theta) = 2 \cot(90 - \alpha) - 1 \cdot \cot(90 - \alpha)$$

$$3 \tan \theta = \tan \alpha$$

$$\tan \theta = \frac{1}{3} \tan \alpha$$



For equilibrium of AB , taking moment about B

$$R \times 3a \sin(90 - \alpha) - W a \cos \theta = 0$$

$$3R \cos \alpha = W \cos \theta$$

$$R = \frac{W \cos \theta}{3 \cos \alpha}$$

$$R = \frac{W}{3 \cos \alpha} \cdot \frac{3}{\sqrt{9 + \tan^2 \alpha}}$$

$$R = \frac{W}{\sqrt{9 \cos^2 \alpha + \sin^2 \alpha}} = \frac{W}{\sqrt{8 \cos^2 \alpha + 1}}$$

$$\sec^2 \theta = \frac{9 + \tan^2 \alpha}{9}$$

$$\cos \theta = \frac{3}{\sqrt{9 + \tan^2 \alpha}}$$

Resolving along the rod

$$S \cdot \cos(90 - \alpha) - R \cdot \cos(90 - \alpha) - W \cdot \cos(90 - \theta) = 0$$

$$S \cdot \sin \alpha - R \cdot \sin \alpha - W \sin \theta = 0$$

$$S \cdot \sin \alpha = R \sin \alpha + W \sin \theta$$

$$= \frac{W}{\cos \alpha} \frac{\sin \alpha}{\sqrt{9 + \tan^2 \alpha}} + W \frac{\tan \alpha}{\sqrt{9 + \tan^2 \alpha}}$$

$$= \frac{2W \tan \alpha}{\sqrt{9 + \tan^2 \alpha}}$$

$$S = \frac{2W}{\cos \alpha \sqrt{9 + \tan^2 \alpha}}$$

$$= \frac{2W}{\sqrt{8 \cos^2 \alpha + 1}}$$

Example-11:

A right circular solid cone of weight W , semivertex angle 30° , and radius of the base a is placed on a smooth inclined plane which makes an angle α to the horizontal. One end of an inextensible string of the length $\sqrt{3}a$ is attached to the centre of the base of the cone and the other end is connected to the inclined plane. If the system is in equilibrium with curved surface in contact with the plane

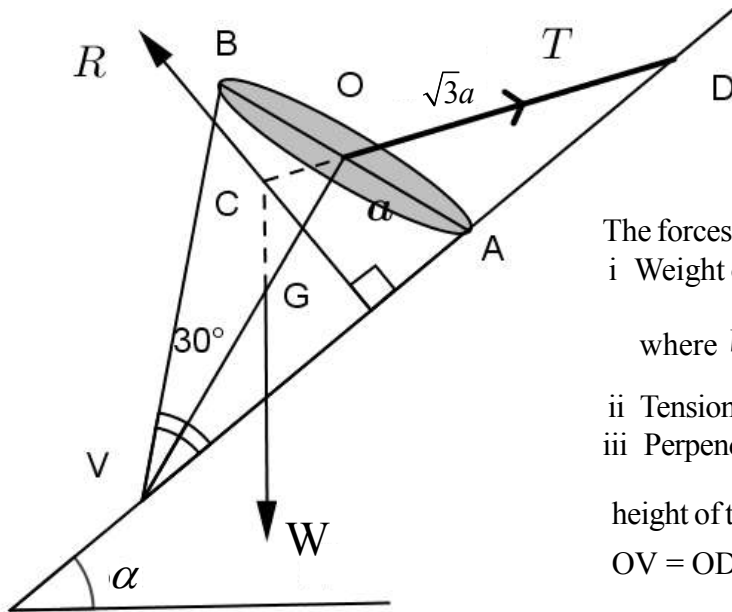
i Show that the tension in the system is $\frac{2\sqrt{3}W \sin \alpha}{3}$

ii Find the reaction between the curved surface and the plane

iii Also show that the line of the reaction cuts the symmetric axis of the cone at a distance

$$\frac{3a}{4} \left[\frac{3\sqrt{3} \cos \alpha + 5 \sin \alpha}{3 \cos \alpha + \sqrt{3} \sin \alpha} \right] \text{ from its vertex.}$$

(The centre of gravity of a solid cone of height h is at a distance $\frac{3h}{4}$ from the vertex)



- The forces acting on the cone are
- Weight of the cone W through G ,
 - Tension in the string
 - Perpendicular reaction of the smooth plane

$$\text{height of the cone} = a \cot 30^\circ = a\sqrt{3}$$

$$OV = OD \Rightarrow \angle ODA = 30^\circ$$

For equilibrium, Forces W and T meet at C so reaction R should pass through C
Resolving parallel to the plane

$$\circlearrowleft T \cos 30 - W \sin \alpha = 0$$

$$T = \frac{2\sqrt{3}W \sin \alpha}{3}$$

Resolving perpendicular to the plane

$$\mathcal{M} R - W \cos \alpha - T \sin 30 = 0$$

$$R = \frac{T}{2} + W \cos \alpha = \frac{W}{3} \left[\sqrt{3} \sin \alpha + 3 \cos \alpha \right]$$

Taking moment about V

$$\mathcal{M} R \times x \cos 30 - W \cdot \frac{3}{4} a \sqrt{3} \cos(30 + \alpha) - T \sin 30 \times 2a \sqrt{3} \frac{\sqrt{3}}{2} = 0$$

$$R \cdot x \frac{\sqrt{3}}{2} = \frac{3\sqrt{3}W}{4} \left[\frac{\sqrt{3}}{2} \cos \alpha - \frac{1}{2} \sin \alpha \right] + \frac{1}{2} \frac{2\sqrt{3}W}{3} \sin \alpha \times 3a$$

$$R \cdot x = \frac{3Wa}{4} (\sqrt{3} \cos \alpha - \sin \alpha) + 2a \sin \alpha$$

$$= \frac{3W}{4} (3\sqrt{3} \cos \alpha - 3 \sin \alpha + 8 \sin \alpha) a$$

$$x = \frac{W}{4} \left(\frac{3\sqrt{3} \cos \alpha + 5 \sin \alpha}{3 \cos \alpha + \sqrt{3} \sin \alpha} \right) \frac{3a}{W}$$

$$x = \frac{3a}{4} \left(\frac{3\sqrt{3} \cos \alpha + 5 \sin \alpha}{3 \cos \alpha + \sqrt{3} \sin \alpha} \right)$$

4.6 Equilibrium under the action of more than three forces

Now we have to consider the general case where there are more than three coplanar forces acting on a rigid body. The forces need not meet at one point.

Any system of forces acting in one plane upon a rigid body can be reduced to a single force R or a single couple G

If $R=0$ its components in any direction is zero.

But $R^2 = X^2 + Y^2$ implies $R=0$ means $X = 0$ and $Y = 0$

ie, The sum of the components in two perpendicular direction each must be zero.

Moment of the couple is the same about any points in its plane we show that if G is zero sum of the moments of the forces about any point in its plane is zero.

Condition for Equilibrium which is sufficient to ensure the equilibrium

- i The sum of the components of the forces in any two direction must be zero and
- ii The algebraic sum of the moments of the forces about any point in their plane is zero.

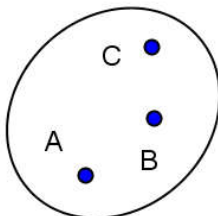
Condition (i) ensures that the system does not reduce to a single force and (ii) ensures not reduces to a couple

Another equivalent conditions

The algebraic sum of moments of the force about any three points in its plane not all in a straight line must be zero.

Proof

Total sum of the moments about A or B is zero means there may be resultant and AB its line of action and sum of moment about C is zero. implies there is no such a force.



4.7 Worked examples

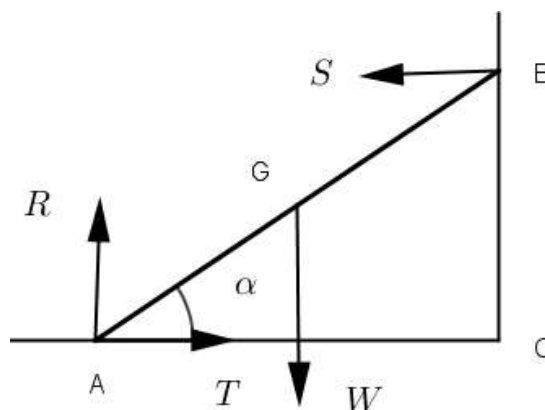
Example 1

A uniform ladder rests at an angle α to the horizontal with its ends resting on a smooth floor and against a smooth vertical wall. The lower end being joined by a string to the junction of wall and floor. Find the tension of the string and the reaction of the wall and the ground.

Find also the tension of the string when a man of equal weight of the ladder has as centered the ladder three quarters of its length.

Four forces acting on the ladder

- i Weight W
- ii Tension of the string T
- iii Reaction of the floor R
- iv Reaction at wall S



For equilibrium of AB

$$\rightarrow T - S = 0$$

$$T = S \dots \dots \dots (1)$$

↑ For equilibrium of AB

$$R - W = 0$$

$$R = W \dots \dots \dots (2)$$

Taking moment about A ∩

$$S \times 2a \sin \alpha - W \times a \cos \alpha = 0$$

$$S = \frac{W}{2} \cot \alpha, T = \frac{W}{2} \cot \alpha$$

When the man is on the ladder

$$R - 2W = 0$$

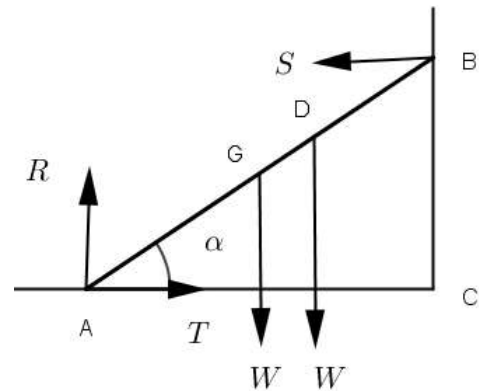
$$R = 2W$$

Moment about B ∩

$$\uparrow T \times 2a \sin \alpha - R \times 2a \cos \alpha + W \times a \cos \alpha + W \times \frac{a}{2} \cos \alpha = 0$$

$$2T \sin \alpha = 2 \times 2W \cos \alpha - \frac{3}{2}W \cos \alpha$$

$$T = \frac{5W \cos \alpha}{4 \sin \alpha} = \frac{5W}{4} \cot \alpha$$



Example 2

A beam of weight W is divided by its centre of gravity G into two portions AC and BC whose lengths are a and b . The beam rests in a vertical plane on a smooth floor AD and against a smooth vertical wall DB . A string is attached to a hook at D and to the beam at a point P . If T is the tension of the string and θ, ϕ be the inclination of the beam and string to the horizontal respectively.

Show that
$$T = \frac{Wa \cos \theta}{(a+b) \sin(\theta - \phi)}$$

For equilibrium of AB

$$\rightarrow T \cos \phi - S = 0$$

$$S = T \cos \phi$$

Taking moment about A

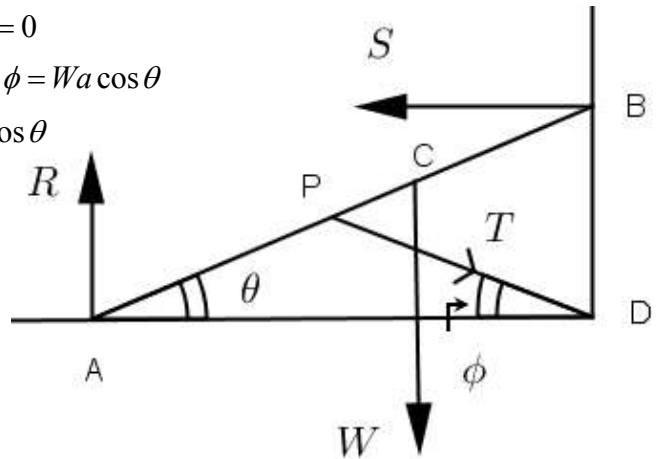
$$m S \times AB \sin \theta - T \times AD \sin \phi - W \times a \cos \theta = 0$$

$$T \cos \phi (a+b) \sin \theta - T \times (a+b) \cos \theta \sin \phi = Wa \cos \theta$$

$$T(a+b)[\sin \theta \cos \phi - \cos \theta \sin \phi] = Wa \cos \theta$$

$$T(a+b) \sin(\theta - \phi) = W \times a \cos \theta$$

$$T = \frac{Wa \cos \theta}{(a+b) \sin(\theta - \phi)}$$



Example 3

To the end B of a uniform rod AB of weight W is attached a particle of weight w. The rod and a particle are suspended from a fixed point O by two light strings OA, OB of the same length as the rod. Prove that in equilibrium position, if T_1 and T_2 are the tensions in strings OA, OB then

$$(i) \frac{T_1}{T_2} = \frac{W}{W + 2w}$$

$$(ii) \text{ If } \alpha \text{ is the angle OA makes with vertical } \tan \alpha = \frac{(W + 2w)\sqrt{3}}{3W + 2w}$$

The forces acting are

- * weight of the rod W
- * weight of the particle w
- * Tensions in the strings T_1 and T_2

The resultant of the parallel forces W and w is a like parallel force through D of magnitude W+w where $GD:DB = w : W$

Let $AB = 2a$ then $GB = a$

$$GD = \frac{w}{W + w} a$$

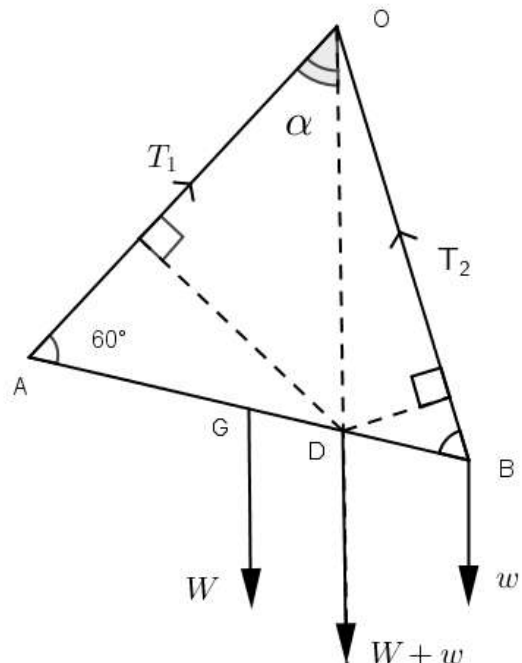
$$AD = a + \frac{wa}{W + w} = \left(\frac{W + 2w}{W + w} \right) a \quad \text{and}$$

$$DB = a - \frac{wa}{W + w} = \frac{Wa}{W + w}$$

Now we have three forces in equilibrium taking moment about D

$$n T_1 \times AD \sin 60 - T_2 \times DB \sin 60 = 0$$

$$\frac{T_1}{T_2} = \frac{DB}{AD} = \frac{Wa}{W + w} \times \frac{W + w}{(W + 2w)a} = \frac{W}{W + 2w}$$



Using sin rule in triangle OAD

$$\frac{AD}{\sin \alpha} = \frac{OA}{\sin [180 - (60 + \alpha)]}$$

$$\frac{AD}{OA} = \frac{\sin \alpha}{\sin (60 + \alpha)}$$

$$\frac{W + 2w}{W + w} \times \frac{a}{2a} = \frac{\sin \alpha}{\frac{\sqrt{3}}{2} \cos \alpha + \frac{1}{2} \sin \alpha}$$

$$\frac{W + 2w}{2(W + w)} = \frac{1}{\frac{\sqrt{3}}{2} \cot \alpha + \frac{1}{2}} = \frac{2}{\sqrt{3} \cot \alpha + 1}$$

$$\frac{\sqrt{3} \cot \alpha + 1}{2} = \frac{2(W + w)}{W + 2w}$$

$$\sqrt{3} \cot \alpha = \frac{4W + 4w}{W + 2w} - 1 = \frac{3W + 2w}{W + 2w}$$

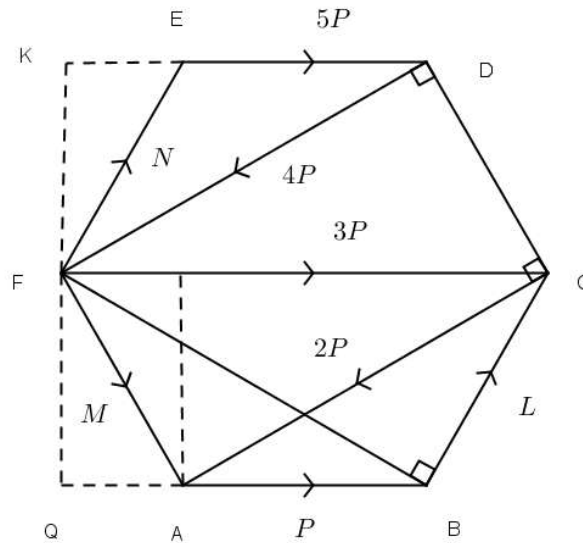
$$\cot \alpha = \left(\frac{3W + 2w}{W + 2w} \right) \frac{1}{\sqrt{3}}$$

$$\tan \alpha = \left(\frac{W + 2w}{3W + 2w} \right) \sqrt{3}$$

Example 4

The Points A,B,C,D,E,F are the vertices of a regular hexagon ABCDEF of side $2a$ makes taken in anticlockwise sense. Forces of magnitudes P,2P,3P,4P,5P,L,M,N newtons act along AB,CA,FC,DF,ED,BC,FA and FE respectively in the direction indicated by the order of the letters. If the system is in equilibrium find L,M,N interms of P.

Given that forces are in equilibrium their sum of moments about any point is zero.



$$FB \perp BC$$

$$FD \perp DC$$

Taking moment about F

$$\sum L \times FB - 5P \times FK + P \times FQ - 2P \times FA = 0$$

$$L \times 4a \cos 30 - 5P \times 2a \sin 60 + P \times 2a \sin 60 - 2P \times 2a = 0$$

$$4L \times \frac{\sqrt{3}}{2} - 10P \times \frac{\sqrt{3}}{2} + 2P \times \frac{\sqrt{3}}{2} - 4Pa = 0$$

$$2\sqrt{3}L - 4\sqrt{3}P - 4P = 0$$

$$L = \frac{4P + 4\sqrt{3}P}{2\sqrt{3}}$$

$$= 2P \left(1 + \frac{1}{\sqrt{3}} \right) N$$

Taking moment about A

$$\sum L \times 2a \cos 30 - 5P \times 4a \cos 30 - N \times 2a \cos 30 - 4P \times 2a - 3P \times 2a \cos 30 = 0$$

$$2L \times \frac{\sqrt{3}}{2} - 2N \times \frac{\sqrt{3}}{2} - 26P \times \frac{\sqrt{3}}{2} + 8P = 0$$

$$L - N - 13P + \frac{8P}{\sqrt{3}} = 0$$

$$N = L - 13P + \frac{8P}{\sqrt{3}}$$

$$= 2P + \frac{2P}{\sqrt{3}} + \frac{8P}{\sqrt{3}} - 13P$$

$$N = \left(\frac{10}{\sqrt{3}} - 11 \right) P$$

Resolving parallel to AB

$$\rightarrow L \cos 60 + M \cos 60 + N \cos 60 + 5P + 3P + P - 4P \cos 30 - 2P \cos 30 = 0$$

$$\frac{L}{2} + \frac{M}{2} + \frac{N}{2} + 9P - 4P \frac{\sqrt{3}}{2} - 2P \frac{\sqrt{3}}{2} = 0$$

$$L + M + N = 6\sqrt{3}P - 18P$$

$$M = 6\sqrt{3}P - 18P - \left(\frac{12P}{\sqrt{3}} - 9P \right)$$

$$= 2\sqrt{3}P - 9P$$

$$= (2\sqrt{3} - 9)P$$

Therefore

$$L = \left(1 + \frac{1}{\sqrt{3}} \right) PN$$

$$M = (2\sqrt{3} - 9) PN$$

$$N = \left(\frac{10}{\sqrt{3}} - 11 \right) PN$$

4.8 Exercises

- (1) A uniform bar AB of weight $2w$ and length l is free to turn about a smooth hinge at its upper end A, and a horizontal force is applied to the other end B so that the bar is in equilibrium with B is at a distance a from the vertical through A. Prove that the reaction at the hinge is

$$\text{equal to } w \left[\frac{4l^2 - 3a^2}{l^2 - a^2} \right]^{\frac{1}{2}}$$

- (2) A uniform rod, of length a hangs against a smooth vertical wall being supported by means of a string of length l , tied to one end of the rod, the other end of the string being attached to a point in the wall. Show that the rod can be inclined to the wall at an angle θ given by

$$\cos^2 \theta = \frac{l^2 - a^2}{3a^2}$$

What are the limits of the ratio of $a : l$ for which equilibrium is possible.

- (3) A sphere of radius r and weight W rests against a smooth vertical wall, in which is attached a string of length l fastened to a point on its surface. Show that the tension in the string is

$$\frac{W(l+r)}{\sqrt{l^2 + 2lr}}$$

Also find the reaction between wall and sphere.

- (4) A solid cone of height h and semi vertical angle α , is placed with its base against a smooth vertical wall is supported by a string attached to the vertex and to a point on the

wall. Show that the greatest possible length of the string is $h\sqrt{1 + \frac{16}{9}\tan^2 \alpha}$

- (5) A triangular lamina ABC is suspended from a point O by light string fastened to points A and B and hangs so that BC is vertical. Prove that if α and β be the angles which strings AO and BO makes with vertical then $2 \cot \alpha - \cot \beta = 3 \cot \beta$

- (6) A uniform rectangular lamina board rests vertically in equilibrium with its side $2a$ and $2b$ on two smooth pegs in the same horizontal line at a distance c apart. Prove that the side of length $2a$ makes with horizontal an angle θ given by $c \cos 2\theta = a \cos \theta - b \sin \theta$
Deduce that a square of side $2a$ will rest on the smooth pegs when the inclination of the

side to the horizontal is as $\frac{1}{2} \sin^{-1} \left(\frac{a^2 - c^2}{c^2} \right)$

- (7) A uniform rod of weight W rests with its ends in contact with two smooth planes inclined at an angle α and β respectively to the horizontal and intersecting in a horizontal line. If θ be the inclination of the rod to the vertical show that $2 \cot \theta = \cot \beta - \cot \alpha$ also find the reaction at the ends.
- (8) A smooth peg is fixed at a point P at a distance a from a smooth vertical wall. A uniform rod AB of length $6a$ and weight W is in equilibrium resting on the peg with the end A in contact with the wall. Taking θ be the angle made by the rod AB with the horizontal draw a triangle of force, representing forces acting on the rod. Find the reaction at P in terms of W and θ , show that $3 \cos^3 \theta = 1$
- (9) A thin rod of length a is in equilibrium with its ends resting on the inner smooth surface of a smooth circular hoop of radius a , fixed its plane vertical. If the centre of gravity divides its length in the ratio 3:4. Prove that the inclination of the rod to the vertical is $\tan^{-1} \left(\frac{7}{\sqrt{3}} \right)$
Determine the ratio of the reaction on the lower end of the rod to that on upper end.
- (10) Two uniform smooth spheres of radius a and weight W lie at rest touching each other inside a fixed smooth hemispherical bowl of radius b ($> 2a$). Draw in separate diagrams, a triangle of forces representing forces acting on the spheres and show that the reaction between the two spheres
is $\frac{Wa}{\sqrt{b(b-2a)}}$
- (11) One end of a uniform beam of weight W is placed on a smooth horizontal plane, the other end to which a string is fastened, rests against another smooth inclined plane, inclined at an angle α to the horizontal. The string passes over a pulley at the top of the inclined plane, hangs vertically and supports a weight P . Show that in equilibrium $2P = W \sin \alpha$
- (12) A rod is movable in a vertical plane about a hinge at one end and at the other end is fastened a weight equal to half the weight of the rod. This end is fastened by a string of length l to a point at a height c vertically above the hinge. Show that the tension in the string is $\frac{lW}{c}$
where W is the weight of the rod.
- (13) $ABCDEF$ is a regular hexagon. Five forces each equal to P act along AE, ED, DC, CB, BA . Five forces each equal to Q act along AC, CE, EB, BD, DA respectively indicated by the order of letters. Prove that the ten forces will be in equilibrium if P and Q are in a certain ratio and find the ratio?